2

nel-lógica

newton c. a. da costa

Notas de aula: lógica e fundamentos da ciência

(organizado por décio krause)

NEL

Notas de Aula

Lógica e Fundamentos da Ciência

SÉRIE NEL-LÓGICA, VOL. 2

NEWTON C. A. DA COSTA

Notas de Aula
Lógica e Fundamentos da Ciência

Organizado por Décio Krause

NEL – Núcleo de Epistemologia e Lógica
GLFC – Grupo de Lógica e Fundamentos da Ciência – UFSC/CNPq
Universidade Federal de Santa Catarina
Florianópolis, 2019

ISBN: 978-85-87253-39-2 (papel)
978-85-87253-40-8 (e-book)

UFSC, Centro de Filosofia e Ciências Humanas, NEL
Caixa Postal 476
Bloco D, 2 andar, sala 209
Florianópolis, SC, 88010-970
(48) 3721-8612
nel@cfh.ufsc.br
http://nel.ufsc.br

FICHA CATALOGRÁFICA
Catalogação na fonte pela Biblioteca Universitária
da
Universidade Federal de Santa Catarina

E64

da Costa, Newton C. A.
Notas de Aula [recurso eletrônico] : Lógica e Fundamentos da Ciência / Newton C. A. da Costa ; Décio Krause (org.). - Florianópolis : NEL/UFSC, 2019.
135 p. - (Série NEL-Lógica ; vol. 2)

Inclui bibliografia.

ISBN 978-85-87253-39-2 (papel)
ISBN 978-85-87253-40-8 (e-book)

1. Lógica - Filosofia 2. Ciência - Filosofia. I. Krause, Décio. II. Série.

CDU: 16

Elaborado pelo bibliotecário Jonathas Troglio – CRB 14/1093

Para Neusa, amor eterno.

The truth of a theory is in the mind, not in the eyes.
Albert Einstein

Sumário

Prefácio

Senti-me muito honrado quando o Professor Newton da Costa me convidou para prefaciar o presente volume. Conheço o Prof. Newton, doravante, e com o devido respeito, simplesmente "Newton", desde a década de 1970, quando fiz um breve curso de lógica com ele (em 1975). Depois de 1984, passei a frequentar suas aulas na USP, onde fiz o meu doutorado sob a sua orientação, e desde então temos contato quase que diário. Newton é um grande exemplo, exemplo de um cientista de primeiro porte, cuja preocupação principal é o empreendimento intelectual e a absoluta dedicação ao trabalho científico. Além disso, seus trabalhos nos campos da lógica (lógicas clássica e não-clássicas, teoria dos modelos), teoria da computação, fundamentos da física, filosofia da ciência e vários outros assuntos, fez dele um cientista internacionalmente reconhecido.[1] Várias pessoas (e isso também aconteceu comigo) me relataram que, no exterior, quando falamos sobre a filosofia que se faz no Brasil (não acredito que haja sentido em se falar de uma "filosofia brasileira"), indagam sobre Newton. Certamente é, no campo da filosofia, nosso filósofo mais conhecido mundialmente.

Os textos que compõem o presente volume são notas de aula ministradas no curso de pós-graduação em filosofia da UFSC, onde Newton está como professor colaborador desde 2003. Sempre digo aos meus alunos que assistir os seus seminários, interagir com ele, aprender com ele, tomar café com ele, são oportunidades que eles dificilmente poderão emparelhar aqui em nosso meio. Temos, com efeito, excelentes professores e filósofos mas, na minha opinião, nenhum do porte de Newton. Além disso, sua capacidade crítica sempre afiada, desafiando os estudantes a procurarem soluções por eles

[1]No primeiro capítulo deste volume, nosso autor comenta sobre algumas áreas nas quais trabalhou ou vem trabalhando.

mesmos aos variados problemas que apresenta, como muitas das questões deixadas ao leitor no que se segue, são o maior ensinamento da necessidade de se buscar o desenvolvimento do necessário tirocínio crítico e da independência de julgamento, termos que ele gosta de usar, aparentemente inspirado em um de seus filósofos favoritos, Bertrand Russell, sem o que ninguém pode sequer começar a caminhar pela estrada da filosofia. Importante mencionar que ele está sempre aberto às tentativas de respostas dadas pelos alunos, com quem adora "discutir".

Os textos refletem o seu estilo presente de não se ater a um assunto específico ao longo de todo o curso, mas de proporcionar ao estudante uma visão de conjunto da lógica e dos fundamentos de parte da ciência presente, notadamente da matemática e da física, com especial relevo sendo dado à física quântica. Os textos são, muitos deles, *handouts* que apresentam as ideias a serem desenvolvidas em sala de aula, dentro da característica de Newton, sempre em tom provocativo e de incentivo. Ele procura salientar assuntos que em geral não fazem parte do quotidiano dos alunos, mostrando a eles que há muitas, verdadeiramente muitas, coisas das quais um filósofo que não se atenha a trivialidades deveria ao menos ter consciência, ainda que, se necessário, deva procurar pelos detalhes. Mas, pelo menos, não se estará ignorando uma enormidade de tópicos relevantes.

Nos escritos que o leitor encontrará à frente, não há um excesso de equações, mas apenas as necessárias para que se possa apresentar as sutilezas filosóficas que jazem na ciência presente, e que ele procura motivar, sempre chamando a atenção para o fato de que soa enganador alguém pretender fazer filosofia da ciência sem dominar minimamente alguma área científica. A exata compreensão das equações nem é tão relevante para o entendimento de suas ideias; o importante é que se saiba que há relações, e que qualquer discussão mais séria sobre os assuntos abordados, requer que se entenda a contraparte matemática, sem o que nada passará de especulação vazia em certo sentido. Por isso ele tanto chama a atenção para o que deve ser estudado, e fornece a direção a ser seguida, lembrando sempre que o seu papel, assim como o de qualquer professor, é o de apontar o Norte; chegar lá é a tarefa do estudante, o que requer esforço e dedicação; afinal, é o estudante quem (por hipótese) quer chegar ao Norte. Repetições de alguns assuntos aparecem em alguns capítulos, mas o leitor deve entender que, como dito acima, tratam-se de notas de aula, e a insistência em temas relevantes é essencial para o aprendizado.

Da mesma forma deve-se considerar que alguns capítulos são escritos em inglês, que se tornou a linguagem padrão na divulgação científica e filosófica. Acredito que não se pode mais prescindir de um conhecimento básico dessa língua para qualquer empreendimento que pretenda alcançar um público mais amplo em qualquer dessas áreas. Por isso, não há mais sentido em realizar traduções dessa língua para a nossa.

Gostaria de agradecer ao NEL e a seus responsáveis presentes, particularmente aos meus colegas Jaimir Conte, Jonas Arenhart e Cezar Mortari, assim como a meus alunos de pós-graduação, pela ajuda inestimável na elaboração deste volume. As falhas são de minha responsabilidade.

Décio Krause
Florianópolis, Agosto de 2019

Parte I

Lógica e Filosofia

1

PHILOSOPHY OF LOGIC

> As a lightning clears the air of unpalatable vapors, so an incisive paradox frees the human intelligence form the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice.
> J. J. Sylvester

I summarize, in the remarks below, some of my views on the philosophy of logic. However, taking into account the limits of space that I have at my disposal, my remarks are schematic; details may be found in the references.

I

Since I was fifteen or sixteen years old, I became interested in the problem of knowledge, its meaning, scope and limits. I immediately perceived how I should proceed to really understand the problem of scientific knowledge: to dedicate myself to certain areas of the scientific domain, in order to get a first hand acquaintance of the foundations of logic, mathematics and a particular field of the empirical sciences. Among these sciences, I have chosen to concentrate on physics. Clearly, the essential role in my endeavour had to be represented by logic, deductive as well inductive, that constitutes the theoretical basis of all sciences, including mathematics. On the other hand, a good amount of general philosophy and, in particular, of epistemology, would be required. Amongst other things, I studied various contemporary philosophers, like Russell, Carnap, Quine, Enriques and Popper. The influence of French thinkers on my philosophical development was also very important, deserving mention Descartes, Brunschvicg, Cavailès and Lautman.

Mathematics, philosophy and physics showed me the relevance of non-classical logics. For example, intuitionistic logic is indispensable to the comprehension of constructive mathematics, and quantum mechanics presents logical difficulties that have suggested the creation of quantum logics. Even though one may defend the thesis that heterodox logics are epistemologically irrelevant, one has to discuss the status of most new logics. However, several problems of general philosophy and also of the philosophy of science almost impose on us the consideration of some new logics as true logics, such as tense logic and paraconsistent logic. But it is clear that other non-classical logics, like non-structural logics, nonlinear logic and the logic of default, are only tools associated to questions of information processing, i.e., to computing, at least today.

We may say, exaggerating a little bit, that the present situation, in the field of logic, forces us to recognize that it is occurring a revolution in that field analogous to that of the birth of non-Euclidean geometries in the XIX century.

Moreover, certain other aspects of the history of logic in the last century also helped to push me to the study of logic and its philosophy; for example, logic that was, since the time of Aristotle till the beginning of XX century, a source of mathematically trivial results, became, little by little, a very complex mathematical edifice. Topics like forcing, forking, Gödel's theorems, algebraic logic, and various other subjects pointed out that logic could be mathematically significant, giving rise to fundamental inquiries and results as profound as the most important mathematical ones. These developments originated new conceptual and significant problems in the spheres of logic, its applications and philosophy.

To confirm my observations on the progress in the domain of logic, it is sufficient to quote from Yu. I. Manin's excellent book *A Course in Mathematical Logic for Mathematicians*, Springer, 2010, page VII:

> In the intervening three decades [since 1977], a lot of interesting things happened to mathematical logic: (i) Model theory has shown that insights acquired in the study of formal languages could be used fruitfully in solving problems of conventional mathematics. (ii) Mathematics has been and is moving with growing acceleration from the set-theoretic language of structures to the language and intuition of (higher) categories, leaving behind old concerns about infinities: a new view of foundations is now emerging. (iii) Computer science, a no-nonsense child of the abstract computability theory, has been creatively dealing with old challenges and providing new ones, such as the P/NP problem.

Independently of the total correctness of Manin's assertions, they demonstrate, at least in outline, that there are occurring profound changes in logic in the last decades.

II

What could I say about my contributions to the philosophy of logic? To begin with, I insist on the fact that any conceptions related to the philosophy of logic are inseparable from my logic work. Therefore, some discussion of the later is essential to explain the former.

Leaving aside some incursions in the domains of induction and tense logic, deontic and modal logics, the logics of belief and of justification, and algebraic logic, my main areas of research in logic are the following: 1) Paraconsistent and non-reflexive logics; 2) Quasi-truth; 3) Generalized Galois theory and its applications to the foundation of physics; 3) The theory of valuations; 5) Recursion theory and complexity.

There is in [38] a good description of the nature of paraconsistent logic; its authors write that, on pages 791 and 792:

> In a few words, paraconsistent logics (PL) are the logics of inconsistent but nontrivial theories. A deductive theory is paraconsistent if its underlying logic is paraconsistent. A theory is inconsistent if there is a formula (a grammatically well-formed expression of its language) such that the formula and its negation are both theorems of the theory; otherwise, the theory is called consistent. A theory is trivial if all formulas of its language are theorems. Roughly speaking, in a trivial theory "everything" (expressed in its language) can be proved. If the underlying

> logic of a theory is classical logic, or even any of the standard logical systems like intuitionistic logic, inconsistency entails triviality, and conversely. So, how can we speak of inconsistent but nontrivial theories? Of course, by changing the underlying logic to one which admits inconsistency without making the system trivial. Paraconsistent logics do just this job.

Our use of terms like 'consistency', 'inconsistency', 'contradictory' and similar ones is syntactical, which is in accordance with the original metamathematical terminology of Hilbert and his school. In order to treat such terms from a semantic point of view, in the field of paraconsistency, one must be able to build, first, a paraconsistent set theory. This is possible, as we will see, although most semantics for paraconsistent logics are classical, i.e., constructed inside classical set theories. So, to begin with, it is best to employ the above terms syntactically.

Inconsistencies appear in various levels of discussion of science and philosophy. For instance, Peirce's world of 'signs' (which we inhabit) is an inconsistent and incomplete world. Bohr's theory of the atom is one of the well-known examples in science of an inconsistent theory. The old quantum theory of black-body radiation, Newtonian cosmology, the (early) theory of infinitesimals in the calculus, the Dirac δ-function, Stokes analysis of pendulum motion, Michelson's 'single-ray' analysis of the Michelson-Morley interferometer arrangement, among others, can also be considered as cases of inconsistencies in science. Given cases such as these, it seems clear that we should not eliminate a priori inconsistent theories, but rather investigate them. In this context, paraconsistent logics acquire a fundamental role within science itself as well as in its philosophy. As we will see below, due to the wide range of applications which nowadays have been found for these logics, they have an important role in applied science as well.

Paraconsistent logic contributed to make clear that the domain of rationality has a wider scope than the traditional one: logic may cope with contradiction without annihilation of the discourse. Although negation inside a paraconsistent logical system cannot be the classical, nonetheless it possesses most formal properties of classical negation, deserving to be considered as a different kind of negation. Strong systems of paraconsistent logic exist and even a paraconsistent mathematics is being developed. In addition, there are various applications of paraconsistent logic, inclusive to real situations, such as those of robotics, distribution of energy, and traffic control in large cities. Today, if one is willing to systematize all natural sciences in a single axiomatic system, this is only possible by the means of a paraconsistent logic, since, for example, quantum mechanics and general relativity are incompatible. (By the way, in some applications the paraconsistent systems are employed as mere technical classical tools to organize inference.) Other references on paraconsistent logic are the following: [21], [23], [34] and [36].

Non reflexive logic was born to cope with two basic kinds of questions: those of causal inference and those originated by topics of quantum mechanics. I quote the following passage connected with this type of logic:

> It is a striking feature of research in logic in the twentieth century that a plurality of logics has emerged. A similar pattern can be found in the case of several logics. What was initially taken to be an unchallenged principle of classical logic turned out to be open to revision in light of the introduction of a suitable logic. For example, excluded middle, an important principle in classical logic, does not hold in general in intuitionistic logic or in many-valued logic. The principle of non-contradiction, another central principle of classical logic, need not hold in paraconsistent logics, but this still leaves open the principle of identity. This principle is challenged by non-reflexive logics.
>
> Roughly speaking, non-reflexive logics are logics in which the principle of identity does not hold in general. One of the main motivations for the construction of these logics (and it turns out that are infinitely many of them) emerges from the foundations of physics. According to certain interpretations of quantum mechanics (such as the one favoured by Schrödinger) it does not make sense to attribute identity to quantum particles. If this is indeed the case, the principle of identity seems to fail. ([35], p. 182)

Some authors call the law that any proposition implies itself the principle of propositional identity. One of the categories of non-reflexive logic includes logics with implications that don't satisfy this principle, also named reflexive law of implication. For example, causal implication and explanatory implications are non-reflexive.

Quasi-truth and generalized Galois theory (or general theory of set-theoretic structures) constitute the basis for a structural sistematization of science, in special of physics. Quasi-truth generalizes Tarki's conception of truth, and its logic is paraconsistent, what should be expected, since theories such as general relativity and quantum mechanics are incompatible. So, the logic of quasi-truth may be seen as the starting point for an axiomatic reconstruction of present-day empirical sciences (see [37], [39], [41] and [88]).

The theory of valuations deals with the subject of a general semantics for the logical systems. It includes the standard Tarskian semantics, but appeals to the syntactic level of the language employed. All logics are complete and correct in relation with their semantics of valuations. In various cases, for instance that of Brouwer-Heyting propositional logic, the method of valuations provides us with new decision methods (see [23] and [38]).

My work on recursion theory and complexity allows us to determine some new limits of the axiomatic method in connection with the foundations of the natural sciences, as well as of mathematics (some references are [42] and [20]).

The sciences aspire objective knowledge. It seems that logic (and its philosophy) entails that scientific knowledge is a kind of belief that is justifiably quasi-true, i.e., objective in certain sense. Paraconsistent and non-reflexive logics are the starting point of some of my views related to the philosophy of logic.

As I noted, in physics of today two of its main theories, quantum theory and general relativity, are logically incompatible. In addition, all physical theories are approximate, and even if true according with the correspondence theory of truth, we would be unable to know that it is so. Therefore, in the foundations of the field of the empirical sciences, we need a new version of truth, that I called quasi-truth, and that, among other things, generalizes Tarski's definition of truth; it consists of a sort of *as if* truth. The most interesting aspect of it is that the logic of quasi-truth is paraconsistent (see [37] and [39]).

Non reflexive logic makes evident that logic has connections with experience, i.e., it isn't entirely a priori. Quantum theory, for instance, drives us to the necessity of a profound analysis of some heterodox logics. I believe that one of the characteristics of contemporary logic is the edification and discussion of new logics, in particular of heterodox logics.

Paraconsistent and non-reflexive logics motivated a profound analysis of logical concepts such as those of negation, conjunction, quantification, and property, inclusive that of set or class. In particular, various groups of negation are possible and convenient.

To summarise, my principal contributions to the philosophy of logic are the following:

1. The analysis of the meaning of non-classical logics, in particular for science, with emphasis on physics.

2. The investigation of negation and other logical concepts taking into account their roles in non-classical logics.

3. The introduction of the concept of quasi-truth and its logic as a new basis for scientific theories (it can be applied even in mathematics, especially in the domain of the so called experimental mathematics).

4. A peculiar approach to inductive logic (which, among other things includes statistics) based, in part, on subjective probability (details in [39] and [40]. In this item, my work on recursion theory and complexity was important to make explicit some limitations essentially involving the scientific method (compare with [42]).

5. The philosophical elucidation of various semantic notions by the means of the theory of valuations, above all the notions of correctness and of completeness of a logical system.

III

Philosophy of logic is mainly the critical study of the foundations of logic and of its signification for the human knowledge in general. In consequence, one of its principal tasks is to contribute to the better understanding of the mathematical and of the scientific knowledge. In this sense, logic helps the philosopher in his critical analysis of different domains of knowledge and to their systematization via the axiomatic method.

Let us take a look in two examples: a) In quantum mechanics there exist various central problems, including that of the nature and meaning, as well as of the scope, of the relation of identity in connection with quantum objects. Non reflexive and non-distributive logics were born to aid us to explain some difficulties involving identity in the area of quantum particles. b) Some forms of metaphysics, for instance structural metaphysics, are related to the logical and mathematical theory of set-theoretic structures (see [42] and [41]). In philosophy of logic we are concerned with those and similar subjects.

The relevance of logic and its philosophy is very well understood with reference to pure mathematics. In applied mathematics, say in information processing and practical computing, which gave origin to several new logics, I had to submit those logics to a detailed philosophical investigation; the signification of such theme becomes clear when we see that it has links to some of the most influential disciplines of our time such as robotics, Artificial Intelligence and neuroscience.

We live in the era of computing, information theory and robotics. Progress in some present-day fields of knowledge is exponential, and, to have a deep understanding of what is occurring we need, surely, to appeal to logic and its philosophy.

IV

In the history of logic numerous significant advances could be here listed and discussed, such as those associated to the names of Aristotle, Boole, Frege and Russell. Notwithstanding, restricting myself to the last one hundred years, I think that the main advances in the philosophy are the following:

1. The philosophical developments motivated by the works of Gödel and Tarski.

2. The philosophical results originated from the non-classical logics, above all the heterodox logics.

3. The inquiries on the logical foundation of quantum mechanics.

4. The definition of quasi-truth and the investigation of its logic.

A few comments on such advances are in order, as we make in what follows. Gödel made a revolution in logic and in the foundations of mathematics. This fact is not only a consequence of his incompleteness theorems, but a result of his various investigations in these domains. Deserves mention his philosophical treatment of the notion of finitary method and of the construtivity of mathematical procedures, his inquiries in the foundations of set theory, and his philosophical reflexions on the meaning of the incompleteness theorems. On this last topic, Gödel says that,

> It is this theorem [the second incompleteness theorem] which makes the incompletability of mathematics particularly evident. For, it makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover I believe that they contain all of mathematics. If somebody makes such statement he contradicts himself. For if he perceives the axioms under consideration to be correct, he also perceives (with the same certainty) that they are consistent. Hence he has a mathematical insight not derivable from the axioms. (K. Gödel, *Collected Works*, vol. III, p. 309.)

The notion of truth is one of the central ideas underlying mathematics and the sciences, as well as logic. The greatest merit of Tarski, from the philosophical point of view, is that he formulated a rigorous concept of truth. In certain sense, he mathematized a notion which always caused problems to the expert devoted to the philosophy of mathematic and of the sciences in general. Tarski's version of truth fits like a glove the intuitive, informal concept of classical truth, constituting a remarkable advance in both logic and epistemology. Better, it was the starting point of numerous extensions of the concept of truth, extensions that, in some cases, are more adapted to the treatment of various questions related to the natural sciences, like the idea of quasi-truth.

I did already make reference to the significance of the birth of non-classical logics, especially of alternative logics. Thus, intuitionistic logic and its variant versions contributed to the clarification of the meaning of constructivity in mathematics and of information processes linked, say, to Turing machines. Paraconsistent logic showed the existence of diverse forms of weak negation and the possibility of a new treatment of contradiction. Some systems of tense logic yield to a deeper philosophical analysis of the concept of time. Non-reflexive logic is going to give rise to novel ideas in the philosophy (details in [56]).

V

The philosophy of logic presents a cluster of important open problems. To cope with this question I shall be limiting myself to what I am doing at the moment, i.e., the researches in which I am presently involved.

One of the principal open problems I am studying is related to the genuine meaning of non-classical logics, in particular of heterodox logics. Have such logics an epistemological significance, are they authentic logics, or are they just mathematical tools founded on classical logic? I believe that the future progress of knowledge, above all of quantum mechanics, will clear up this question. However, we need to learn much more of quantum mechanics, quantum logic and related matters, to surmount the difficulty.

Another subject that must be philosophically explained is the relation between logic and space-time. The common systems of logic are built independently of considerations of space and time; but how is this legitimately possible, if logic is structured as if there exists neither space nor time, and is applied to objects that are in space and time? In fact we have various space-times in theoretical physics: for example, the Newtonian (of classical mechanics and of non-relativistic quantum mechanics), the one of Minkowski (of special relativity and of quantum field theory) and the space-times of general relativity (of quantum field theory in curved spaces and those of cosmology). Then, what are the precise links between space-time and logic? To cope with this question, a philosophical examination of the bases of logic and of contemporary physics is essential.

A third interesting problem is the following: I conceive logic as a discipline incorporating what is usually called inductive logic, but in which statistics is included. Today, a serious philosophical difficulty with respect to this part of logic consists in the following: the presupposed concept of probability, needed by non-relativistic quantum mechanics, is not the common one of standard probability theory and statistics. In effect, in quantum mechanics there isn't any Kolmogorov basic structure, since the set of events don't satisfy the conditions required by the custumary theory of probability. In addition, the standard rule of conditionalization doesn't work. In order to develop quantum mechanics, as well as most of its interpretations, should we need a new theory of probabilities and a new statistics? The philosophical investigations of this topic are elucidating it, and effective progress is expected.

I have just listed the three preceding problems mainly because I think that logic is not a subject entirely justifiable by itself; on the contrary, its value is, in a great part, consequence of its connections with the world of mathematics and of science, in particular of physics, and also with everyday inferences. So, its main philosophical relevance depends on issues involving, in part, other regions of knowledge.

Anyhow, I would like to emphasize that progress, in philosophy, means above all a better understanding of the problems examined. As Bertrand Russell would say, the goal of philosophy is to replace inarticulate beliefs by articulate disbeliefs.

2

ALTERNATIVISMO CRÍTICO (RESUMO)

> Não sou jovem o suficiente para saber tudo.
> Oscar Wilde

PROCURAREMOS, neste trabalho de índole expositiva, delinear nossa posição em filosofia da ciência, que denominaremos de *alternativismo* ou *pragmatismo crítico*. Esta posição baseia-se nas seguintes considerações, que procuramos defender em nossas obras (por exemplo em [22] e [23]):

1. No tocante à lógica: há diversas lógicas em princípio possíveis. A aceitação de uma delas, em certa disciplina científica, depende de fatores basicamente pragmáticos, no sentido, por exemplo, de Carnap, Morris e especialmente [22]. A teoria geral da linguagem ou semiografia engloba duas dimensões centrais: a metassemiótica e a semiótica, a primeira concernente a princípios filosóficos e segunda de índole científica; esta subdivide-se em sintática, semântica e pragmática (para detalhes, ver [22] e [23]).

   O núcleo da ciência consiste, sobretudo, nas teorias científicas em vigor, ou seja, na sua parte linguística, inclusive pragmática. Concebemos esta última como englobando, entre outros tópicos, toda a metodologia científica.

2. No tocante às várias teorias científicas, se uma delas salva as aparências em determinada área da ciência, em geral muitas outras, bem diversas, também dão conta das mesmas situações empíricas (caso óbvio da mecânica quântica não relativista).

3. As divergências na aceitação de teorias em disciplinas científicas, tais como a física e a química, depende, às vezes muito, de fatores metassemióticos e pragmáticos.

   A análise informal e crítica dos pressupostos da ciência é tarefa fundamental: de fato, o exercício do tirocínio crítico, em parte independente de qualquer sistematização teórica, afigura-se imprescindível para se fazer filosofia da ciência. Tal atividade pode ser batizada de análise filosófica informal e se enquadra na metassemiótica.

4. As concepções metafísicas são pragmaticamente legítimas, embora não se disponha de nenhuma prova efetiva de que qualquer delas seja verdadeira, nem que se imponha que uma dentre elas se mostre mais ou menos a legítima. Em todo caso, a metafísica ajuda muito na elaboração das teorias científicas e até pode unificá-las de um ponto de vista geral. No entanto, ela se situa além das possíveis verificações positivas ...

5. O exercício do tirocínio crítico permanente é essencial em ciência. Não há, no entanto, critério geral que nos permita diferenciar ciência de metafísica, muito embora em casos extremos isto seja mais ou menos patente.

6. Uma das necessidades da ciência é a de se dever procurar testar experimentalmente as teorias científicas, excluindo-se lógica e matemática. Sem possibilidade de confirmação experimental, não há ciência fatual.

7. Lógica e matemática são ciências, em princípio, não empíricas. Entre outras finalidades, são utilizadas na edificação rigorosas das teorias científicas.

8. Uma teoria é estritamente científica se puder, ainda que em princípio, ser verificada ou testada. Não obstante, há teorias que não são científicas neste sentido, as quais, não obstante, podem auxiliar a ciência: elas merecem, por assim dizer, a designação de teorias auxiliares da ciência e as batizamos de *quase-teorias* ou de *teorias quase-científicas*. Isto ocorre, por exemplo, com teorias como a do Big Bang e a da evolução, que têm características muito especiais, em especial ajudando a tarefa dos cientistas. (Estas considerações podem soar um tanto estranhas, mas, afirmou Joseph Pérès, "A ciência começa onde o senso comum termina.")

9. Essencial é o conceito de justificação ou de verificação, envolvendo teorias da medicão, dos erros e a estatística, englobando o cálculo de probabilidades. A ciência empírica é feita modulo uma teoria da justificação. Modificando-se esta última, transforma-se o sentido da ciência experimental . . .

10. As teorias quase-científicas se mostram básicas em todas as areas da ciência, sobretudo em disciplinas tais como a economia e a psicologia.

11. Acreditamos que Gonseth tem razão ao afirmar: "A ciência não marcha de verdade em verdade e de realidade em realidade, porém de horizonte de verdade em horizonte de verdade e de horizonte de realidade em horizonte de realidade." Isto se passa sem aparentemente dispormos de recursos indubitáveis para se encontrar a verdade e a realidade. Bertrand Russell, certa vez, fazendo graça, escreveu: "I would never die for my beliefs, since I could be wrong." De nosso ponto de vista, a filosofia da ciência mais conveniente é, então, um *alternativismo* ou um *pragmatismo crítico*, implícitos em tudo o que asseveramos anteriormente e nos nos três parágrafos que se seguem.

12. O esquema que acabamos de delinear pode ser expandido e completado. Não parece que aqui valha a pena entrar em minúcias para se defender uma filosofia da ciência tal como foi resumida acima. O essencial não é tratar de pormenores, muitos dos quais se encontram em nossas obras, porém tão somente apresentar um esquema, um mapa de nossa maneira de encarar a ciência. Se o mapa for apropriado, ele deixará

claro os traços gerais de nossa posição, sem termos de mergulhar até o fundo da problemática em tela, tão funda que parece não ter fundo. Se o mapa não for conveniente, surge de imediato a tarefa de se obter outro . . . O filósofo da ciência é, por assim dizer, uma espécie de cartógrafo que sempre busca aperfeiçoar, pouco a pouco, suas habilidades de cartógrafo, permanentemente buscando mapas que cada vez nos orientem melhor, aprimorando nosso conhecimento e nos permitindo a confecção de novos mapas, mais e mais confiáveis.

13. Um ponto digno de nota, embora trivial, é que a filosofia da ciência não deve ser elaborada sem se ter um certo conhecimento da própria ciência; no tocante às ciências naturais, é fundamental que isso ocorra independentemente de se possuir uma certa familiaridade com a lógica e a matemática. Mais ainda, uma boa filosofia da ciência precisa poder contribuir pra o próprio desenvolvimento da ciência como um todo ou de alguma ciência em particular, ainda que se limite a uma contribuição tão somente conceitual ou crítica.

14. A atividade científica apresenta também dimensões sociológica e econômica, as quais se mostram básicas para se compreender a ciência em geral. Essas se constituem, em essência, produto social que possui aspecto econômico muito significativo. Sem entrar em detalhes sobre o prisma sociológico, lembremos que a face econômica é clara, por exemplo na física hodierna, na qual a aparelhagem científica e suas instalações dependem de fatores econômicos e politicos significativos. Toda essa discussão mostra que a pragmática é central para o estudioso da ciência, caso deseje entender bem sua natureza. Mesmo a metassemiótica é significativa nesse sentido. A ciência possui uma dimensão humana que lhe é nuclear.

1. Notas para reflexão do leitor

I

À lógica usual pode-se conferir uma versão algébrica, por exemplo como foi feito por Halmos. Por curiosidade, vejamos como em tal lógica algébrica pode-se enunciar os teoremas de completude e de incompletude de Gödel:

[Teorema da completude do cálculo de predicados de primeira ordem] Qualquer álgebra poliádica localmente finita de grau infinito é semanticamente completa.

[Teorema de incompletude] Qualquer álgebra de Peano (que é uma algebra poliádica) é sintaticamente incompleta.

Qual o significado desses resultados? (Cf. Paul R. Halmos, *Algebraic Logic*, Chelsea, 1962.)

II

Uma descoberta básica sobre a mecânica quântica não relativista é o seguinte teorema de Kochen-Specker: *Não se podem atribuir valores, simultaneamente, a todas as quantidades quânticas de um sistema físico, desde que sejam mantidas as relações existentes entre tais grandezas, impostas pela mecânica quântica (não relativista), supondo-se que que a dimensão do espaço de Hilbert correspondente seja maior ou igual a três.*

Uma consequência desse teorema é a de que as proposições quânticas, relativas a um sistema quântico, em cada momento do tempo, não podem, todas elas, serem verdadeiras ou falsas. Em cada instante do tempo, sempre haverá proposições quânticas que não têm valor lógico de verdade ou de falsidade. Ou seja, a lógica da mecânica quântica não pode ser a clássica.

Será que a ultima sentença do período acima é verdadeira? O que seria uma sentença quântica em semelhante contexto? Uma resposta pode ser dada nas seguinte linha: uma sentença quântica é uma sentença formulada na linguagem da mecânica quântica, isto é, expressável, em princípio, por meio da análise funcional e da contraparte empírica da MQ. De fato, parece que não existe uma definição totalmente precisa de tal conceito. Aliás, no referente a este assunto, poder-se-ia defender, hoje, a seguinte tese: *na ciência experimental não existe precisão completa como em matemática.*

III

Haverá outras teorias científicas que necessitem, provadamente, de lógicas não clássicas? Este seria o caso de uma teoria da física mais forte do que a mecânica quântica não relativista? Aqui a questão é a seguinte: logicamente, a MQ é incompatível com a relatividade geral. Como se aceitam as duas, será que a lógica da física não seria paraconsistente?

3

IDENTIDADE E IGUALDADE

> Não exijo que uma teoria corresponda à realidade porque não sei o que ela é. A realidade não é uma qualidade que se possa testar com azul de tornassol.
> Stephen Hawking

O conceito de identidade evoluiu no decurso da história. Por exemplo, já foi visto como permanência na mudança ou como unidade na diversidade. Aos poucos, a identidade passou a ter o seu significado atual. Isto se deu, sobretudo, a partir de Leibniz, quando a identidade adquiriu o significado de relação que só vige entre um objeto e ele mesmo, o que está perfeitamente claro em Frege e em Russell.

O princípio de identidade dos indiscerníveis de Leibniz garante que objetos diferindo apenas pelo número têm que ser idênticos e que objetos idênticos são indiscerníveis (possuem as mesmas propriedades). De fato, o princípio em apreço envolve dois outros: o da indiscernibilidade dos idênticos e o da identidade dos indiscerníveis.

Note-se, no entanto, que o princípio em tela parece nos conduzir a um beco sem saída, pois necessitamos saber quais são as propriedades pertinentes, quando elas são as mesmas, em que condições são idênticas, etc. Uma espécie de reparo como o precedente, somente pode ser superado por meio de uma formulação rigorosa da teoria da identidade como, v. g., uma teoria de tipos lógicos.

Ponto debatido no referente ao princípio de Leibniz radica na natureza das características (propriedades) que sustentam a identidade: seriam suficientes as propriedades ‘intrínsicas’ dos objetos para assegurar sua identidade ou seria preciso se recorrer a propriedades ‘relacionais’ (‘extrínsicas’), tais como as espaço-temporais?

Tais questões se acham no âmago da teoria da identidade. Na sua versão presente, devida sobretudo a Frege e Russell, as propriedades são encaradas de modo amplo, englobando propriedades ‘relacionais’. Obviamente, isto se encontra ligado ao princípio de separação em teoria de conjuntos e ao princípio de redutibilidade de Russell (sobre a identidade, ver, por exemplo, o livro de Mendelson [71].)

Normalmente, a identidade significa ‘ser a mesma coisa’, ou seja, uma noção metafísica, muito utilizada, atualmente, em lógica e em matemática. Ela é uma relação binária, reflexiva, simétrica e transitiva, além de satisfazer uma lei de substitutividade apropriada. A formulação desta lei depende do sistema lógico que se empregar.

Conceito aparentado à identidade é o de relação de equivalência (compatível com as propriedades e relações da linguagem com a qual se está trabalhando) ou, como diremos algumas vezes, de igualdade. Se nossa lógica for clássica ou qualquer outra dentre as comuns, não há como se distinguir, em nível sintático e de modo perfeito, entre identidade e igualdade o que é consequência do metateorema da impossibilidade de distinção entre identidade e igualdade em um sistema dedutivo assentado na lógica usual.

Objetos distintos, não sendo pois idênticos, podem ocupar o mesmo lugar no espaço em dado instante? Em caso afirmativo, isto significaria que tais objetos teriam a possibilidade de estar na mesma posição no espaço-tempo padrão (de Galileo-Newton) Na física macroscópica isso é impossível. No tocante à mecânica quântica não relativista, fundada em tal espaço-tempo, dado que as medidas de espaço e de tempo são sempre intrinsicamente aproximadas, um tanto ‘fuzzy’, a localização de uma partícula elementar, em um pequeno intervalo de tempo, não é bem definida (princípio de Heisenberg: posição e momento, bem como energia e tempo, são grandezas complementares, sujeitas a esse princípio de indeterminação). Porém, partículas quânticas distintas podem sempre, em princípio, se separadas espaço-temporalmente.

Claramente, esta última suposição constitui um postulado, o qual conecta as partículas elementares com o espaço-tempo. Em síntese, o espaço-tempo usual pode distinguir, em certas situações, partículas de mesma natureza, que, portanto, possuem as mesmas propriedades intrínsecas (independentes do espaço-tempo). E algo análogo, no caso de fótons, ocorre na teoria quântica de campos.

Em um agregado de Bose-Einstein ou de Fermi-Dirac conhecemos os números de repetição de partículas, embora suas posições não sejam bem determinadas. Em certo sentido, parece que se perdem as identidades das partículas, elas ficando indistinguíveis. Assim, em certos sistemas quânticos, 'perdem-se' as identidades das partículas componentes.

Todavia, há fenômenos nos quais a identidade de partículas é mantida e clara: é o caso, por exemplo, da partícula Priscilla nas investigações revolucionárias de Dehmelt (ver [104]). Priscilla foi presa e isolada de qualquer outra, por Dehmelt e sua equipe, em uma armadilha de campos, sendo observada por ele e seus colaboradores, bem como por visitantes de seu laboratório, durante um longo período. No transcorrer da experiência, verificou-se que a proposição 'Priscilla = Priscilla' se manteve obviamente verdadeira.

Em síntese, em MQ (mecânica quântica não relativista) embora haja relação de identidade, com partículas quânticas sendo efetivamente idênticas, inúmeras vezes tudo se passa como se, por assim dizer, houvesse objetos sem identidade.

Tudo isso foi confirmado e estendido posteriormente por Haroche e Wineland que mostraram, entre outras coisas, como se pode controlar partículas quânticas em geral ([104] e [95]).

Hans Dehmelt ganhou o Prêmio Nobel em 1989 e Serge Haroche e David Wineland receberam tal prêmio em 2012.

Nota: Recentemente, Dominik Zumbühl e Daniel Loss desenvolveram muito nosso conhecimento das partículas elementares.

Loss afirmou:

"to put it simply, we can use this method to show what an electron looks like for the first time". E Zumbühl deixa claro que : "We are able to not only map the shape and orientation of the electron, but also control the wave function according to the configuration of the applied electric fields. This gives us the opportunity to optimize control of the spins in a very targeted manner."[1]

[1]Ver: https://physorg/news/2019-05-geometry-electron.html ("the geometry of the electron determined for the first time", by the University of Basel.)

4

INTRODUCTION TO ABSTRACT LOGICS

THIS CHAPTER contains the first pages of a monography on Abstract Logics, yet to be typed in full. It is added here for it introduces the basic notions and with the hope be useful for the reader to consider the first notions, yet it is still incomplete.

1. Introduction

The main objective of the present work is the study of the concept of an *abstract logic*, which is a kind of structures that generalizes most of the extant logics. We develop the subject with a view to its inclusion into the domain of generalized Galois theory (see da Costa [30]), considered as a general theory of structures.

Some authors have already treated similar logical structures, for example Wójcicki ([106] and [107]), as well as various of the authors cited in these books, and Béziau ([11]). However, their approaches to an abstract formulation of logical structures are different from ours, though they can be adapted to our special form of developing the theme. In this connection, Tarski deserves special mention, since his calculus of systems constitutes one of our major influences (see Tarski [99]).

A logical system is usually based on axioms and rules; they make up the starting point to define the concept of (formal) theory. Sometimes, instead of axioms and rules of inference, the notion of semantic consequence also constitutes the basis of the concept of theory (this is the case, for example, of the model-theoretical logics).

With theories at hand, we are able to characterize the consequence relation, the logical operators, etc. So, we take as fundamental the concept of theory. The theories of a logical system form a generalized topology; therefore, abstract logics will be here conceived as certain generalized topological spaces.

It commonly happens that a logic has a specific underlying language. Thus, most of our abstract logics are linked to free algebras, more precisely to what we call free pre-algebras, because languages can normally be envisaged as such pre-algebras.

In summary, we will be concerned with certain generalized topological spaces, similar to the standard ones, named abstract logics. In most cases, they are defined on free pre-algebras.

2. Abstract logics

An abstract logic is a structure of the following form:

$$\ell = \langle F, \mathcal{T} \rangle \tag{4.1}$$

where $F \neq \emptyset$ is called the set of formulas and $\mathcal{T}$, such that $\mathcal{T} \neq \emptyset$ and $\mathcal{T} \subset \mathbb{P}(F)$, $\mathbb{P}(F)$ being the power set of F, is the set of theories, satisfying the postulates:

1. $F \in \mathcal{T}$
2. If $(A_i)_{i \in I}$ is a family of elements of $\mathcal{T}$, then $\bigcap_{i \in I} A_i \in \mathcal{T}$

We call F the initial theory, and note that the second postulate express a fundamental property of the most usual logical systems.

Formulas will be denoted by small Greek letters and sets of formulas by capital Latin letters.

Definition 4.1. Given the set of formulas $A \cup \{\alpha\}$, we write $A \vdash \alpha$ if any theory T containing A is such that $\alpha \in T$; $\vdash$ constitutes the consequence relation of ℓ, and $\vdash \alpha$ abbreviates $\emptyset \vdash \alpha$. $A \nvdash \alpha$ means that it is not the case that $A \vdash \alpha$.

Theorem 4.2. *In ℓ:*

1. $\alpha \in A \Rightarrow A \vdash \alpha$
2. $A \vdash \alpha \Rightarrow A \cup B \vdash \alpha$
3. *If $A \vdash \alpha$ for every $\alpha \in B$ and $B \vdash \beta$, then $A \vdash \beta$*
4. *If $(A_i)_{i \in I}$ is a family of subsets of F, such that $\forall\alpha(\alpha \in A_i \Leftrightarrow A_i \vdash \alpha)$, then $\forall\alpha(\alpha \in \bigcap_{i \in I} A_i \Leftrightarrow \bigcap_{i \in I} A_i \vdash \alpha)$*

Theorem 4.3. *If T_A (or $\overline{\mathrm{A}}$) designates the intersection of all theories containing A, then $A \vdash \alpha \Leftrightarrow \alpha \in \mathrm{T}_A$.*

Theorem 4.4. *In ℓ:*

$$T \in \mathcal{T} \Leftrightarrow \forall\alpha(\alpha \in T \Leftrightarrow T \vdash \alpha).$$

Definition 4.5. $\{\alpha : \vdash \alpha\}$ is the set of (logically) valid formulas of ℓ.

Theorem 4.6. $\{\alpha : \vdash \alpha\} = \bigcap \mathcal{T}$.

When $\bigcap \mathcal{T} = \emptyset$, ℓ has no valid formulas.

Definition 4.7. The automorphisms of ℓ are the bijections of F that leave $\mathcal{T}$ invariant. The set of such bijections with the operation of composition of functions form the Galois group of ℓ.

The generalized Galois theory as developed, for instance, in [] and [], clearly includes the abstract logics. So, if two abstract logics $\ell_1 = \langle F, \mathcal{T}_1 \rangle$ and $\ell_2 = \langle F, \mathcal{T}_2 \rangle$ have the same Galois group, then $\mathcal{T}_2$ is definable in the wide sense in terms of $\mathcal{T}_1$ and conversely. Moreover, the structures in consideration compose the species of structures of abstract logic in the terminology of Bourbaki ([13]) or of axiomatic or mathematical system.

Definition 4.8. $\overline{A}$ designates $\{\alpha : A \vdash \alpha\}$ ($\overline{\ }$ is Tarski's consequence operator, commonly represented by Cn, but introduced in a different context).

Theorem 4.9. *In ℓ, one has:*

1. $\overline{F} = F$
2. $A \subset \overline{A}$
3. $\overline{\overline{A}} \subset \overline{A}$
4. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
5. *5) If $(A_i)_{i \in I}$ is a family of subsets of F and, for all $i \in I, A_i = \overline{A_i}$, then* $\overline{\bigcap_{I \in I} A_i} = \bigcap_{i \in I} A_i$.

Definition 4.10. A is deductively closed if $A = \overline{A}$; A is deductively open when A is the complement of a deductively closed set. (When there is no danger of confusion,we suppress the adverb 'deductively').

Theorem 4.11. *The intersection of any collection of closed sets is closed. Any closed set is a theory and conversely. If $\{\alpha : \vdash \alpha\} \neq \emptyset$, then an open set cannot be a theory. The theories, with obvious operations, form a lattice.*

Theorem 4.12. *Let $\ell' = \langle F, \Vdash \rangle$ be a structure named abstract consequence logic, such that $F \neq \emptyset$ and $\Vdash$ is a relation between subsets of F and elements of F, satisfying the conditions:*

1. $\alpha \in A \Rightarrow A \Vdash \alpha$
2. $A \Vdash \alpha \Rightarrow A \cup B \Vdash \alpha$
3. *If $A \Vdash \alpha$ for every element α of B and $B \Vdash \beta$, then $A \Vdash \beta$*

Then, there is one and only one manner to transform ℓ' in an abstract logic $\ell = \langle F, \mathcal{T} \rangle$ whose consequence operator $\vdash$ coincides with $\Vdash$; in order to get ℓ, it is enough to define $\mathcal{T}$ as follows: $\mathcal{T} = \{T : \forall\alpha(\alpha \in T \Leftrightarrow T \vdash \alpha\}$.

Proof. Firstly we show that the set $\mathcal{T}$ satisfies the postulates of an abstract logic. In effect, $F \in \mathcal{T}$ by clause 1 above and by the fact that all formulas belong to F. By the definition of $\mathcal{T}$, if A_i, $i \in I$, is a family of subsets of F belonging to $\mathcal{T}$, then $A = \bigcap_{i \in I} A_i \in \mathcal{T}$. For, if $A \Vdash \alpha$, then $A_i \Vdash \alpha$, $i \in I$, and $\alpha \in A_i$, $i \in I$; hence, $\alpha \in A$. But if $\alpha \in A$, then clearly $\vdash \alpha$, and $A \in \mathcal{T}$. Now, we observe that defining $\vdash$ by the condition

$$A \vdash \alpha \Leftrightarrow \forall T \in \mathcal{T}(A \subset T \Rightarrow \alpha \in T),$$

we have that $\Vdash = \vdash$. In fact, one has in ℓ' that

$$A \vdash \alpha \Leftrightarrow \forall T \in \mathcal{T}(A \subset T \Rightarrow \alpha \in T),$$

what is immediate to prove. On the other hand, it is true that in ℓ' we have that

$$A \vdash \alpha \Leftrightarrow \forall T \in T(A \subset T \Rightarrow \alpha \in T).$$

Therefore, $\Vdash = \vdash$. Finally, since $\vdash$ is univocally determined by the set $\mathcal{T}$, this is the only manner to transform ℓ' in ℓ. □

Analogously, given an abstract logic $\ell = \langle T, \mathcal{T} \rangle$, there is only and only manner to transform ℓ in an abstract consequence logic $\ell' = \langle F, \vdash \rangle$, such that its set of theories is precisely $\mathcal{T}$. Thus, the species of structures of abstract logic and abstract consequence logic are equivalent (Bourbaki [13]).

Since there exist various species of structures equivalent to topological space (employing the -primitive- notions of neighborhood, open set, interior, filter, etc.), there are also several species of structures equivalent to that of abstract logic. Some of them seem to be interesting to our objectives.

3. Maximal sets

Let us consider an abstract logic $\ell = \langle F, \mathcal{T} \rangle$. We put:

Definition 4.13. A set $A \subset F$ is said to be non-trivial (not dense) if $\overline{A} \neq F$; otherwise, A is trivial or dense. A is maximal non trivial if A is non trivial and is not properly contained in another non trivial set.

Non triviality is a generalization of consistency, a very important notion of classical logic.

Definition 4.14. A is an α-theory (or α-closed) if A is a theory, $A \nvdash \alpha$ and $\forall\beta(A \nvdash \beta \Rightarrow A \cup \{\beta\} \vdash \alpha)$. An α-theory is maximal if it is not properly contained in another α-theory.

Theorem 4.15.

1. *Any maximal non trivial set is a theory.*
2. *A is maximal non trivial entails that there is α such that α is an α-theory.*
3. *A is an α-theory entails that A is a maximal α-theory.*
4. *If A is such that $A \nvdash \alpha$ and $\forall\beta(A \nvdash \beta \Rightarrow A \cup \{\beta\} \vdash \alpha)$, then A is an α-theory.*
5. *There exist α-theories that are not maximal non trivial theories.*

Theorem 4.16. *A is a maximal non trivial set iff for all α and β it is true that $A \nvdash \alpha \Rightarrow A \cup \{\alpha\} \vdash \beta$, and there exists γ such that $A \nvdash \gamma$.*

Proof. If A is maximal and non trivial, then there exists γ such that $A \nvdash \gamma$. Moreover, under the hypothesis that $A \nvdash \alpha$, it follows that $A \cup \{\alpha\} \vdash \beta$ for any β, taking into account that A is maximal non trivial. Conversely, let us admit that $A \nvdash \gamma$; hence, A is non trivial. Now, if for all α and β, $A \nvdash \alpha$ implies that $A \cup \{\alpha\} \vdash \beta$, A has to be maximal non trivial: if it were not maximal non trivial, then it would exist B maximal non trivial properly containing A and, as a consequence, there is $\delta \in B$ such that $A \nvdash \delta$; hence, $A \cup \{\delta\} \vdash \beta$, for every β, and B would not be non trivial. □

Theorem 4.17. *A is maximal non trivial iff $\exists\alpha$(A is an theory and $\overline{A \cup \{\alpha\}} = F$).*

The abstract logic $k = \langle F, \sigma \rangle$, where

$$F = \{0, 1, 2, \ldots\} \text{ and } \sigma = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots\},$$

does not possess maximal non trivial sets.

In a logic logic $k' = \langle F, \sigma \rangle$, where F is finite and $\sigma = \mathbb{F}$, any non trivial set of formulas is contained in a maximal non trivial one. The maximal non trivial sets are obtained from F by taking one and only one of its elements out.

In the classical and intuitionistic propositional calculi, the maximal non trivial sets are the maximal consistent sets.

A model of a set of sentences in first-order logic defines a maximal consistent set of sentences, those true in the model. Conversely, any such set of sentences determines a set of models. We are here envisaging, as it is obvious, first-order logic as an abstract logic $\mathcal{L} = \langle F, \sigma \rangle$, F being the set of sentences of first-order logic and σ the set of its theories.

A maximal non trivial set of formulas may be seen as a skeleton of a family of models, at least in the case of several extant logics. However, in the general case it is better to consider α-theories as partial skeletons of models of sets of sentences. This circumstance will become little by little clear and clear.

The preceding consequences motivate the following definition:

Definition 4.18. The abstract logic $\ell = \langle F, \mathcal{T} \rangle$ is perfect if $A \nvdash \alpha$ entails that there is an α-theory B such that $A \subset B$; ℓ is called regular when any non trivial set of formulas of ℓ is contained in a maximal non trivial set.

The logic $k = \langle F, \sigma \rangle$,

$$F = \{0, 1, 2, \ldots\} \text{ and } \sigma = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots\}$$

is regular but not perfect. The classical propositional calculus is perfect and regular.

Another important properties of usual logics are given by the following definition:

Definition 4.19. An abstract logic $\ell = \langle F, \mathcal{T} \rangle$ is compact if, for all A and α, $A \vdash \alpha$ implies that there is $B \subset A$, B finite, such that $B \vdash \alpha$. A logic $\ell = \langle F, \mathcal{T} \rangle$ is finitely trivializable if there exist $a_1, a_2, \ldots, a_n$ satisfying the condition: $\overline{\{a_1, a_2, \ldots, a_n\}} = F$, $n \in \omega$.

The next proposition is fundamental and we owe it essentially to Lindenbaum.

Theorem 4.20. *Let $\ell = \langle F, \mathcal{T} \rangle$ be a logic. Then:*

1. *If ℓ is compact and finitely trivializable, then ℓ is regular.*

2. *If ℓ is compact, then ℓ is perfect.*

Proof. Employing Zorn's lemma. In the first case it is central the class of non trivial theories that contain a certain set; in the second, the class of non trivial theories to which determinate elements does not belong. □

It deserves mention that (L_1) and (L_2) are equivalent to the axiom of choice, as it was shown by Dzik ([50]).

The classical propositional calculus is regular, but the classical implicative propositional logic isn't; however, it is perfect.

Models of sets of sentences are linked to the properties of regularity and perfection. Most logics which are not regular are in fact perfect. These observations motivate the next definition.

Definition 4.21. A valuation is the characteristic function of an α-theory. The valuation υ is a model of A iff $\upsilon(\alpha) = 1$ for every $\alpha \in A$. When any model of A is also a model of $\{\alpha\}$, we write $A \vDash \alpha$ and say that α is a semantic consequence of A. If $A = \emptyset$, instead of $\emptyset \vDash \alpha$, we write $\vDash \alpha$, and α is called valid or a thesis of the logic we are considering.

Theorem 4.22. *If $\ell = \langle F, \mathcal{T} \rangle$ is a perfect abstract logic, one has:*

$$A \vdash \alpha \text{ iff } A \vDash \alpha.$$

A similar theorem can be proved for logics which are regular, finitely trivializable and contain a classical implication. This happens, for instance, with the classical propositional calculus and the modal logics based on classical logic.

4. Logical operators

Any abstract logic may be enriched with operators acting on formulas to give new formulas. Some of the most important of such operators are introduced in the sequel.

4.1. Intuitionistic implication

A binary operator, denoted by $\rightarrow$, is called an intuitionistic implication if it satisfies the following conditions:

1. $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$
2. $A \cup \{\alpha\} \vdash \beta \Rightarrow A \vdash \alpha \rightarrow \beta$

Such definition is, in some sense, *syntactical*. However, we can employ $\vDash$ instead of $\vdash$, obtaining a *semantical* definition.

Theorem 4.23. *The operator $\rightarrow$ is an intuitionistic implication iff one has:*

1. $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$
2. $\vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
3. $\vdash \alpha$ *and* $\vdash \alpha \rightarrow \beta$ *entail that* $\vdash \beta$

4.2. Classical implication

The operator $\rightarrow$ above is a classical implication if it is an intuitionistic implication and

1. $(\alpha \rightarrow \beta) \rightarrow \alpha \vdash \alpha$

Theorem 4.24. *If $\ell = \langle F, \rightarrow, \mathcal{T} \rangle$ is an abstract logic with a classical implication $\rightarrow$, then, for any valuation υ it is true that*

1. $\upsilon(\alpha) = \upsilon(\beta) = 1 \Rightarrow \upsilon(\alpha \rightarrow \beta) = 1$
2. $\upsilon(\alpha) = \upsilon(\beta) = 0 \Rightarrow \upsilon(\alpha \rightarrow \beta) = 1$

3. $\upsilon(\alpha) = 0$ *and* $\upsilon(\beta) = 1 \Rightarrow \upsilon(\alpha \to \beta) = 1$

4. $\upsilon(\alpha) = 1$ *and* $\upsilon(\beta) = 0 \Rightarrow \upsilon(\alpha \to \beta) = 0$

and conversely (truth-table for classical implication).

Theorem 4.25. *Let* $\ell = \langle F, \to, \mathcal{T} \rangle$ *be an abstract logic with a classical implication; then, for any valuation* υ*,*

$$\upsilon(\alpha) = 1 \ \text{or} \ \upsilon(\alpha \to \beta) = 1.$$

4.3. Equivalence

According to S. Leśniewski, classical implication constitutes one of the most important kinds of binary operators. A *classical equivalence*, $\leftrightarrow$ is a binary operator satisfying the clauses below:

1. $\vdash \alpha \leftrightarrow \alpha$
2. $\{\alpha \leftrightarrow \beta\} \vdash \beta \leftrightarrow \alpha$
3. $\{\alpha \leftrightarrow \beta, \beta \leftrightarrow \gamma\} \vdash \alpha \leftrightarrow \gamma$
4. $\{(\alpha \leftrightarrow \beta) \leftrightarrow \gamma\} \vdash \alpha \leftrightarrow (\beta \leftrightarrow \gamma)$
5. $\{\alpha \leftrightarrow \beta, \alpha\} \vdash \beta$
6. $\{\alpha \leftrightarrow \beta, \beta\} \vdash \alpha$

An *intuitionistic equivalence* is characterized by ($\leftrightarrow_1$) to ($\leftrightarrow_4$), without ($\leftrightarrow_4$).

Theorem 4.26. *In an abstract logic with classical equivalence it is true that, for any valuation* υ*:*

1. $\upsilon(\alpha \leftrightarrow \beta) = 1$ *iff* $\upsilon(\alpha) = \upsilon(\beta)$

2. $\upsilon(\alpha \leftrightarrow \beta) = 0$ *iff* $\upsilon(\alpha) \neq \upsilon(\beta)$

The above proposition gives us the truth-table for classical equivalence.

Theorem 4.27. *In a logic with intuitionistic implication (classical implication) $\to$ and equivalence $\leftrightarrow$, one has:*

1. $\alpha \leftrightarrow \beta \vdash \alpha \to \beta$

2. $\alpha \leftrightarrow \beta \vdash \beta$

3. $\{\alpha \to \beta, \beta \to \alpha\} \vdash \alpha \leftrightarrow \beta$

Theorem 4.28. *If a logic possess two intuitionistic (classical) implications, $\to_1$ and $\to_2$, then*

$$\upsilon(\alpha \to_1 \beta) = \upsilon(\alpha \to_2 \beta),$$

for any valuation υ. Similarly for two intuitionistic or classical equivalences.

It is clear that in certain logics with two implications (or equivalences), if one is classical and the other is intuitionistic, then there is a valuation which distinguishes them.

The logic of pure intuitionistic equivalence, that is, a logic whose only (binary) operator is an intuitionistic equivalence, is not decidable by two-valued truth-tables; the same is true in connection with the pure logic of intuitionistic implication. Anyhow, this kind of implication is decidable, as well as the logic of intuitionistic equivalence.

4.4. Disjunction

A disjunction is a binary operator, here denoted by $\vee$, such that:

1. $(\vee_1)$ $\alpha \vdash \alpha \vee \beta$

2. $(\vee_2)$ $\beta \vdash \alpha \vee \beta$

3. $(\vee_3)$ $A \cup \{\alpha\} \vdash \gamma$ and $A \cup \{\beta\} \vdash \gamma$ entails that $A \cup \{\alpha \vee \beta\} \vdash \gamma$

The pure logic of disjunction has no valid formulas (or thesis), but only *valid* deductions, expressions of the form $A \vdash \alpha$. It is decidable by the usual method of two-valued truth-tables.

4.5. Conjunction

A conjunction is a binary operator, here designated by $\wedge$, such that:

1. $(\wedge_1)$ $\{\alpha \wedge \beta\} \vdash \alpha$
2. $(\wedge_2)$ $\{\alpha \wedge \beta\} \vdash \beta$
3. $(\wedge_3)$ $\{\alpha, \beta\} \vdash \alpha \wedge \beta$

The logic of pure conjunction has no valid formulas, only valid deductions. The pure logic of conjunction is decidable by the standard two-valued truth-tables.

4.6. Negation

We consider three types of negations, which are unary operators:

Minimal negation

Minimal negation, or Kolmogorov-Johanson negation, is characterized by the following condition:

1. $(\neg_1)$ $A \cup \{\alpha\} \vdash \beta$ and $A \cup \{\neg\alpha\} \vdash \beta$ implies that $A \vdash \neg\alpha$

Intuitionistic negation

Intuitionistic negation, or Brouwer-Heyting negation, is a minimal negation which satisfies

1. $(\neg_2)$ $\{\alpha, \neg\alpha\} \vdash \beta$

Classical negation

Classical negation satisfies $(\neg_1)$, $(\neg_2)$ and

1. $(\neg_3)$ $A \cup \{\alpha\} \vdash \beta$ and $A \cup \{\neg\alpha\} \vdash \beta$ entails that $A \vdash \beta$.

5. Some of the most usual logics

In this section we shall consider some of the most important logics which can be dealt with within the above framework. The tools discussed till now give raise to propositional logics; they do not contain certain operators named quantifiers. When a logic includes quantifiers, it is a quantificational logic. These logics will be considered in the next section.

To begin with, let us state the following result.

Theorem 4.29. *Let* $\ell = \langle F, \to, \wedge, \vee, \neg, \mathcal{T} \rangle$ *a logic such that:*

1. $\vdash \alpha \to (\beta \to \alpha)$
2. $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3. $\vdash \alpha$ *and* $\vdash \alpha \to \beta$ *entails* $\vdash \beta$
4. $\vdash (\alpha \wedge \beta) \to \alpha$
5. $\vdash (\alpha \wedge \beta) \to \beta$
6. $\vdash \alpha \to (\beta \to (\alpha \wedge \beta))$
7. $\vdash \alpha \to (\alpha \vee \beta)$
8. $\vdash \beta \to (\alpha \vee \beta)$
9. $\vdash (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
10. $\vdash (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$
11. $(\alpha \wedge \neg\alpha) \to \beta$
12. $\alpha \vee \neg\alpha$

then one has that: (a) $\to$ *and* $\neg$ *are classical implication and negation; (b) if (12) is dropped, then* $\to$ *and* $\neg$ *are intuitionistic implication and negation respectively; (c) if (11) and (12) are dropped, then* $\to$ *and* $\neg$ *are intuitionistic implication and minimal negation respectively.*

Definition 4.30. A *classical logic* is a logic like that of the theorem above, which satisfies (1) to (12); a *minimal logic* satisfies (1) to (9) and an *intuitionistic logic* is a minimal logic in which (11) but not (12) is true.

Given a logic, it is possible to define some new operators in terms of the operators of that logic. For example, classical logic can be formulated by means of $\rightarrow$ and $\neg$; $\wedge$ and $\vee$ are obtained by means of the usual definitions. However, this mode of speach is not completely correct: we should say that classical logic with $\rightarrow$ and $\neg$ constitutes a species of structures equivalent to the species of structures called classical logic.

Of course, we could consider a logic of conjunction and disjunction with the help of the postulates of these two operators plus distributivity of one of them in relation to the other.

Definition 4.31. An abstract logic whose only operator is intuitionistic implication (classical implication) is called intuitionistic (classical) implicative logic. The logics governed by conditions (1)-(9) are intuitionistic positive logics; if the postulate $\vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$ (Peirce's law) is added to them, the resulting logics are the classical positive logics.

Theorem 4.32. *Classical implicative logics are strictly stronger than intuitionistic implicative logics. Classical positive logics are also strictly stronger than intuitionistic positive logics.*

Proof. By the method of valuations, i.e., using the properties of the semantic of valuations. □

Definition 4.33. Let ℓ be an abstract logic with operators. ℓ is atomic if its formulas are generated by means of operators and certain simple formulas called atomic formulas. ℓ is finitely trivializable if there exists a formula α such that, in ℓ, $\alpha \vdash \beta$, β being any formula whatsoever. The atomic formulas of ℓ constitute its atoms.

Theorem 4.34. *We assume that ℓ is an atomic positive logic. Then, no one of the negations above introduced are definable in ℓ, if ℓ has an infinite set of atoms. All the foregoing logics, with the exception of intuitionistic and classical logics, are finitely trivializable.*

Proof. By the method of valuations. □

We easily define when a logic is a conservative extension of another.

Theorem 4.35. *Concerning the hierarchy of logics introduced, the addition of a new operator in a logic ℓ originates a conservative extension of ℓ.*

Proof. By the properties of the semantics of valuations of those logics. □

6. Quantificational logics

Now, we pass to take into account quantificational logics, that is, logics with the unary operators called quantifiers. Their formulas contain parts named variables and parts named (non-logical) constants. Normally, the set of variables is denumerable and the set of constants is arbitrary.

A quantifier is a unary operator of the form Qx, where x is the variable attached to it. There are two principal kinds of quantifier: the existential quantifiers $\exists x$, $\exists y$, ..., and the universal quantifiers $\forall x$, $\forall y$, ...; x, y, ... denote variables. If there are infinitely many variables, then there are infinitely many existential and universal quantifiers.

We define in the standard way when an occurrence of a given variable is bound in a formula, when a variable occurs free in a formula, when a variable x is free for a variable y in a formula, etc.

A quantificational logic possesses universal and existential quantifiers (sometimes we may consider abstract logics enriched with only one species of quantifier: the universal or the existential). The postulates governing the quantifiers are listed bellow. But to list them, we need two previous definitions.

Definition 4.36. Retelling the void quantifiers of the formula α is to replace in α any occurrence of quantifiers of the form $QxQ'x$ by $Q'x$.

Definition 4.37. Let f be a bijection of the set of variables on itself. Reteling the variables of the formula α, without free variables, consists in changing each variable x of α by $f(x)$.

Postulates of quantification, where x is a variable, a a nonlogical constant or a variable, and $\alpha(x)$ a formula in which x may occur free (the remaining notations are clear, since we are generalizing the terminology and notations of common predicate logic):

1. (Q1) $\forall x\alpha(x) \vdash \alpha(a)$

2. (Q2) $\alpha(a) \vdash \exists x\alpha(x)$

3. (Q3) If α' is the result of relettering void quantifiers of α, then $\alpha \vdash \alpha'$.

4. (Q4) If α' is obtained from the closed formula α by relettering its variables, then $\alpha \vdash \alpha'$.

When the logic we are considering has implication $\rightarrow$, the postulates (Q1) and (Q2) are replaced by

1. (Q1') $\forall x\alpha(x) \rightarrow \alpha(a)$

2. (Q2') $\alpha(a) \rightarrow \exists x\alpha(x)$

where the standard restrictions of predicate logic are supposed to be satisfied.

If quantifiers are adjoined to an abstract classical logic, the new structure is a classical quantifier logic. Similarly, when quantifiers are adjoined to intuitionistic implicative logic, the logic so obtained is an intuitionistic quantificational implicative logic, etc.

We easily define when a logic is a conservative extension of another. The quantificational logics constitute conservative extensions of the corresponding non quantificational ones. There also exist various relations of conservative extension between the logics previously introduced.

It was presented, above, a generalization of the standard notion of quantifier. However, there are many other ways to make such kind of generalization. Bellow, we shall discuss Krasner's generalization.

We remark that it is possible to study logics with only one type of quantifier, say the existential, and that in some logics one of the quantifiers is definable in terms of the other.

All customary logics can be envisaged as particular cases of abstract logics. For instance, *the* classical propositional calculus (cpc) may assume the form

$$C = \langle F, \vee, \neg, \mathcal{T} \rangle,$$

where F is the set of formulas generated by a denumerably infinite set of propositional variables, $\vee$ is disjunction, $\neg$ is classical negation and $\mathcal{T}$ is the set of all theories based on the calculus.

Obviously, the so to say transformation of *the* classical propositional calculus in an abstract logic may be accomplished in various ways. So, the calculus may be converted into an abstract logic as follows:

$$C' = \langle F', \rightarrow, \neg, \mathcal{T}' \rangle,$$

where $\rightarrow$ and $\neg$ are classical implication and classical negation. The structures C and C' are in effect equivalent or, better, the species of structures corresponding to C and C' are equivalent.

The logical calculus, such as the classical propositional calculus, the classical first-order predicate calculus with or without equality, the intuitionistic propositional calculus and classical monadic first-order calculus, constitute in fact classes of equivalent structures or species of structures; this is what happens with the concepts of topological space, group and differentiable manifold, characteristic of modern mathematics.

7. Applications

In this section we consider, in outline, some applications of the theory of abstract logics.

7.1. The classical propositional calculus (cpc)

The underlying language of cpc is the common one, generated by a set of propositional variables. It may be axiomatized by the postulates (1)-(12) formulated in Theorem 4.29, conveniently adapted. The cpc is essentially an abstract logic of the form

$$C = \langle F, \rightarrow, \wedge, \vee, \neg, \mathcal{T} \rangle.$$

The valid formulas (or thesis or theorems) being the formulas α such that $\vdash \alpha$, and the theories being sets of formulas closed by modus ponens.

In the cpc the implication $\rightarrow$ satisfies the clause

$$\alpha \vdash \beta \rightarrow \alpha.$$

So, for every evaluation υ, we must have $\upsilon(\beta \to \alpha) = 1$, if $\upsilon(\alpha) = 1$, independently of the value of β.

We also have that $\vdash \alpha \vee (\alpha \to \beta)$, therefore, if $\upsilon(\alpha) = \upsilon(\beta) = 0$, then $\upsilon(\alpha \to \beta) = 1$.

In the cpc it is true that $\vdash (\alpha \wedge (\alpha \to \beta)) \to \beta$, what implies that $\upsilon(\alpha) = 1$ and $\upsilon(\beta) = 0$ entails that $\upsilon(\alpha \to \beta) = 0$. In summary, we have established the usual truth-table for $\to$.

Analogously, one can derive the standard truth-tables for conjunction, disjunction, negation and equivalence. Therefore, the valid formulas of the cpc are the tautologies (see theorem below), the maximal non-trivial sets are the maximal consistent sets (the set A is consistent if there is no formula α such that $A \vdash \alpha$ and $A \vdash \neg\alpha$; otherwise, A is called inconsistent), a set is consistent if and only if it is non trivial and the following proposition is provable, which is called the theorem of soundness and completeness of the cpc:

Theorem 4.38. *In cpc, $A \vdash \alpha$ iff $A \vDash \alpha$.*

On the other hand, the cpc is compact. Moreover, when the set of propositional variables is recursive, cpc is decidable (there is an algorithm to verify if a given formula is or not valid, or a tautology).

7.2. Some Lindenbaum-like results

In this subsection it is assumed that $\overline{\overline{F}} = \aleph_0$, i.e., that the set of formulas of the cpc is denumerable. We introduce some concepts and results (without proof) due to Lindenbaum (cf. Tarski [99], chapters III and IV). However the proofs of the results presented offer no substantial difficulty. It will become clear that most of the results can be extended to other abstract logics.

Definition 4.39. Given two sets of formulas A and B, we write $A \sim B$ (A is logically equivalent to B) if $\overline{A} = \overline{B}$. A set X is concistent if $\overline{X} \neq F$; otherwise, X is inconsistent. X is complete if, for every consistent set Y, $X \subset Y$ implies $\overline{X} = \overline{Y}$. (Evidently, X is complete iff $\overline{X} = F$ or $\overline{X}$ is a maximal non trivial set or a maximal consistent set).

Theorem 4.40. *If $\mathcal{R} \neq \emptyset$ is a set of theories such that for every X and Y belonging to $\mathcal{R}$, there exists Z in $\mathcal{R}$, with $Z \subset \mathcal{R}$ and $Y \subset Z$, then $\bigcup \mathcal{R}$ is a theory.*

Theorem 4.41 (of Lindenbaum)**.** *No theory is the union of a finite set of theories distinct from itself.*

Theorem 4.42. *$\mathcal{R}$ is a finite collection of theories Y is a theory, $Y \subset \bigcup \mathcal{R}$ and $Y \neq \emptyset$. Under these hypothesis, there is $X \in \mathcal{R}$ such that $Y \subset X$.*

Definition 4.43. The degree of completeness of a set X, denoted by $\Upsilon(X)$, is the smallest ordinal $\alpha \neq 0$ which satisfies the condition: there is no increasing sequence of sets X, of ordinal α, whose first term is X. (A sequence of sets is increasing if any term is contained in the following.)

Theorem 4.44. $\Upsilon(X) = 1$ *iff* $\overline{X} = F$. $\Upsilon(X) = 2$ *iff X is consistent and complete.* $\Upsilon(X) > 2$ *iff X is consistent but not complete.*

Definition 4.45. X is called independent if $X \sim Y$ and $Y \subset X$ imply that $X = Y$.

Definition 4.46. Y is a basis of X if $X \sim Y$ and Y is independent.

Definition 4.47. Y is a finite axiom system of X (or for short an axioms system for X) if Y is a basis of X and $\overline{\overline{Y}} < \aleph_0$.

Definition 4.48. X is finitely axiomatizable, or for short axiomatizable, if X has a finite basis.

Theorem 4.49. *X is complete iff for every $y \in X$, $(y - \{y\}) \cup \{\neg y\}$ is consistent, where $\neg y$ is the (classical) negation of y.*

Theorem 4.50. *Any finite set of formulas contains a subset which is a basis.*

Theorem 4.51. *The following statements are equivalent: (i) X is complete; (ii) X is finitely axiomatizable; (iii) X has a basis and is finitely axiomatizable; (iv) There is $y \subset X$ such that $\overline{\overline{Y}} < \aleph_0$ and Y is a basis of X.*

Theorem 4.52. *X is independent iff X possesses no infinite basis.*

Theorem 4.53. *Let $\mathcal{T}$ and $\mathcal{A}$ be a class of theories and the class of axiomatizable sets, respectively. Then:* $\overline{\overline{\mathcal{T} \cup \mathcal{A}}} \leq \aleph_0$ *and* $\overline{\overline{\mathcal{T} - \mathcal{A}}} \leq \mathcal{T} \leq 2^{\aleph_0}$. *If there is an infinite set in $\mathcal{A}$, then* $\overline{\overline{\mathcal{T} \cup \mathcal{A}}} = \aleph_0$ *and* $\overline{\overline{\mathcal{T} - \mathcal{A}}} = \overline{\overline{\mathcal{T}}} = 2^{\aleph_0}$.

Proofs of the above results may be founded in Tarski 1983, chapter V. In chapter XII of the same book there are more developments of the subject matter of this section.

Obviously, the theory of abstract logics encompasses Tarski's methodology of deductive sciences.

7.3. Predicate calculi

We assume that the set of formulas is (infinitely) countable and that the same is true with the set of the variables. Moreover, it is assumed that any formula is composed by a finite number of elements (normally symbols), including the variables, each occurring a finite number of times.

When the quantifiers are added to the cpc, with the concept of formula as above, we can define the notion of theory as follows: a theory is a set of formulas closed by *modus ponens* and postulates I1, II1, III and IV, $\forall$ and $\exists$, some of which are *rules* and other *schemes*. The resulting logics are called predicate calculi and all of them are compact. Therefore, the following propositions are provable:

Theorem 4.54. *Any consistent set of formulas has a model.*

Theorem 4.55. $A \vdash \alpha$ *iff* $A \vDash \alpha$.

The preceding statement constitutes the soundness and completeness theorem of the classical first-order predicate calculus in a generalized setting: of course, everything remains valid if the symbol of equality, with appropriate postulates are joined to it. The calculus so obtained is a generalized version of the classical first-order predicate calculus with equality.

Going in the inner composition of formulas, by the inclusion of predicate symbols, etc., as well as developing the semantic of first-order structures, by the Tarskian definition of truth, etc., any valuation determines a corresponding structure and conversely. In this way we get the soundness theorem, Gödel's completeness theorem and the common results of model theory in the standard form.

By the introduction of types, with convenient modifications, we obtain the semantic of Henkin models for higher-order classical logics. Standard models may be considered, but in this case we lose completeness, compactness, etc.

Most statements relative to the classical predicate calculus remain valid for other predicate calculi, for example the classical implicative quantificational logic, the classical positive quantificational logic and the minimal quantificational calculus.

A non trivial theory is said to be irreducible if it is not the intersection of a family of theories different from itself. Otherwise, it is reducible.

Theorem 4.56. *If A is an irreducible, then A is an α-theory.*

Proof. Let $\mathcal{B}$ be the intersection of all theories of the form $A \cup \{\beta\}$, for $\beta \notin A$. Then, if $\beta \in \bigcap \mathcal{B}$, it is easy to see that A is an α-theory. □

Theorem 4.57. *If A is an α-theory, then A sis irreducible.*

Proof. A is an α-theory implies that $A \nvdash \alpha$, i.e., A is non trivial. If A is the intersection of a family of theories different from A, every such theory must contain a β such that $A \nvdash \beta$ and $A \cup \{\beta\} \vdash \alpha$. So, α would be an element of A, and this is absurd. □

Theorem 4.58. *If A is an α-theory and β belongs to the intersection of all theories different from A and containing β, then A is also a β-theory.*

Proof. Consequence of the proof of the theorem 4.56. □

A *cobasis* is a family $\mathcal{B}$ of non trivial theories such that any non trivial theory is the intersection of a subfamily of $\mathcal{B}$. A cobasis $\mathcal{B}$ is minimal when none subfamiliy of $\mathcal{B}$ is also a cobasis. (Every abstract logic has a least a cobasis; however, there are abstract logics without a minimal cobasis.)

Theorem 4.59. *A cobasis is minimal iff its elements are irreducible.*

Theorem 4.60. *Every maximal non trivial set of formulas is irreducible.*

Proof. In effect, any maximal non trivial set is an α-theory for some α. □

Theorem 4.61. *A minimal cobasis is contained in any cobasis and, if it exists, it is unique.*

In the case of compact logics the set of α-theories constitutes a cobasis, in fact a minimal cobasis; in the case of regular logics, the maximal non trivial set form a minimal cobasis. In general, any cobasis is the starting point of the development of a theory of models for a given abstract logic (see [86]).

It deserves to be observed that the standard definition of compact space in general topology is not convenient in connection with abstract logics; in effect, if the intersection of all theories of an abstract logic is not empty, then it would be compact. This is the reason that explains that we didn't choose the standard definition of compacity as a definition of this notion in the theory of abstract logics.

8. Language

An algebraic structure has the form

$$\mathfrak{A} = \langle M, f_k \rangle,$$

where M is a non empty set and f_k a sequence of operations defined on M, each of them with a fixed rank.

If f is a function that to each element of $M^n = M \times \ldots \times M$, with $\times$ meaning the Cartesian product, if assigns one and one element of M, then f is an operation of rank n, or an n-adic operator, on M, $0 \leq n < \omega$. When $n = 0$, the operation f, by extension, determines a fixed element of M. In order to express that f applied to the n-tuple $x_0, x_1, \ldots, x_{n-1}$ gives the value y, we write

$$f(x_0, x_1, \ldots, x_{n-1}) = y.$$

For $n = 2$, it is common to write, instead of $f(x + 0, x_1)$,

$$x_0 \, f \, x_1.$$

1-adic and 2-adic operations are also named *monadic* and *binary* operations, respectively.

Let us suppose that on the set M it is defined a binary relation $\equiv$ (that is, a subset of $M \times M$), satisfying the conditions, for all x, y and z in M:

1. $x \equiv x$
2. $x \equiv y \Rightarrow y \equiv x$
3. $(x \equiv y \text{ and } y \equiv z) \Rightarrow x \equiv z$

that is, $\equiv$ is an equivalence relation on M.

An n-adic operation f is i-compatible with $\equiv$, $0 \leq n \leq n$, iff for any

$$a_0, a_1, \ldots, a_{n-1}, a_i, \ldots, a_{n-1},$$

a and b in M, one has:

$$f(a_0, a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_{n-1}) \equiv f(a_0, a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{n-1})$$

whenever $a \equiv b$.

f is compatible with $\equiv$ iff f is i-compatible with $\equiv$ for every i, $0 \leq i \leq n$.

A *pre-order* on M is a binary relation $\leq$ verifying the conditions, for all x, y and z in M:

1. $x \leq x$
2. $x \leq y$ and $y \leq z$ imply $x \equiv y$
3. $x \leq y$ and $y \leq z$ imply $x \leq z$.

It is then easy to define when an operation is monotone in connection with $\leq$; the operation is nonmonotone in connection with $\equiv$ if it is not compatible with $\equiv$. Easily we define an i-monotone operation related to $\leq$ and $\equiv$ (cf. Curry [18] and [19]).

An n-adic operation, $n \geq 2$, is called idempotent if $f(a, a, \ldots, a) \equiv a$ for every $a \in M$.

The formula $f(a_0, a_1, \ldots, a_{n-1}) = c$ means, as it is obvious, that the value of f at the point $\langle a_0, a_1, \ldots, a_{n-1} \rangle$ is congruent to c.

Let f be an n-adic operation on M and $\equiv$ an equivalence relation on M. If $x_i \equiv y_i$, $i = 1, \ldots, n$, entails that $f(x_1, \ldots, x_n) \equiv f(y_1, \ldots, y_n)$, then $\equiv$ is a *congruence relation* on M in connection with f. Analogously, we define congruence relations in connection with a family of operations on M. If f_λ is a family of operations defined on M and $\equiv$ is an equivalence relation on M

which is a congruence in connection to all operations in f_λ, then we say that $\equiv$ is a *congruence relation* on M.

A structure

$$\varrho = \langle M, \equiv, f_\lambda \rangle,$$

where $\equiv$ is a congruence relation on M and f_λ is a sequence of operations defined on M, is called a *pre-algebra*. Especially in the case in which there are idempotent and incompatible operations in ϱ (that is, equivalence relations which are not congruences on M), we say that the pre-algebra is a Curry algebra.

All standard algebraic structures (semi-groups, groups, fields, rings, Boolean rings, linear algebras, ...) constitute algebras in which $\equiv$ is $=$. In some of them, like Boolean rings, there are idempotent operations.

When in the pre-algebra ϱ all the operations are compatible with $\equiv$, we may pass the quotient and get a new algebraic structure, obtaining a standard algebra. However, if there are incompatible operations, this is not possible.

To make the algebrization of a logical system S is to get an algebra that algebraically shows the nature of the logical properties of S. Given S, the common manner to algebrize S is to find an equivalence relation between formulas of S and afterwards to pass the quotient.

For instance, in the classical propositional calculus, the appropriate equivalence relation is $\vdash \alpha \leftrightarrow \beta$, α and β being any formulas whose logical equivalence is provable. The resulting algebra is a Boolean algebra (in effect, a free Boolean algebra generated by the equivalence classes of propositional variables). Then, by abstraction, one gets the species of structures of Boolean algebras, of which the initial free algebraic structures, originated by passing the quotient, where particular cases.

However, as we shall see, there are logical systems which don't exhibit any significant equivalence relations linking their formulas.

Since from the algebraic point of view they are pre-algebras, one of the possible ways to algebrize them is to treat such systems directly as pre-algebras.

In what follows, *algebra* will designate not only algebras properly speaking, but also pre-algebras. This move will not cause any confusion.

Practically all algebraic versions of the most important logics are lattices.

A good part of the concepts and results of algebra can be extended to pre-algebras even when they present non monotone operations. So, it is not difficult to extend to pre-algebras the concepts of homomorphism, of isomorphism, automorphism, Cartesian product, free algebras, etc., and to prove results such that the second isomorphism theorem and the basic theorems on free algebras.

To explain how all these things can be done, we treat the notion of free prealgebra.

Let $\mathcal{A} = \langle A, \equiv, f_\lambda \rangle$ and $\mathcal{B} = \langle B, \equiv', f'_\lambda \rangle$ be two similar prealgebras, what means that the sequences f_λ and f'_λ have the same domain (which is an ordinal) and the corresponding operations f_λ and f'_λ have the same (finite) rank. Instead of f'_λ and $\equiv'$, we usually write f_λ and $\equiv$. Sometimes, we omit the subscript of f_λ.

Definition 4.62. $\mathcal{A}$ and $\mathcal{B}$ being, as above, prealgebras, the function $h : A \mapsto B$ is called a homomorphism of $\mathcal{A}$ in $\mathcal{B}$ if, for all x, y and y_i in A, one has:

1. $x \equiv y \to h(x) \equiv h(y)$
2. $h((f(y_i)) = f(h(y_i))$

If $|u|$ denotes the equivalence classes determined by u, then

1. $h(|u|) = |h(u)|$
2. $f(x_i) \equiv f(y_i)$ in $\mathcal{A}$ implies that $f(h(x_i)) \equiv f(h(y_i))$ in $\mathcal{B}$
3. $x_i \equiv y_i$ and $h(x_i) \neq h(y_i)$ for some x_i and y_i imply that $[h(f(x_i)) \not\equiv f(y_i)]$.

Definition 4.63. An isomorphism of $\mathcal{A}$ in $\mathcal{B}$ is a bijection between A and B such that

1. $x \equiv y \leftrightarrow f(x) \equiv f(y)$
2. $h(f(x_i)) = f(h(x_i))$
3. $x_i \equiv y_i$ and $\exists x_{i_0} y_{i_0} [h(x_{i_0}) \neq h(y_{i_0}), 1 \leq i_0 \leq i]$ imply that $[f(x_i) \not\equiv f(y_i) \leftrightarrow f(h(x_i)) \not\equiv f(h(y_i))]$

It is obvious that an isomorphism is also an homomorphism.

The homomorphism h of $\mathcal{A}$ in $\mathcal{B}$ can be extended to a function $\widehat{h}$ of the set of equivalence classes of $\mathcal{A}$ into the set of equivalence classes of $\mathcal{B}$. $\widehat{h}$ is the conjugate of h. We say that there exists essentially one homomorphism of $\mathcal{A}$ in $\mathcal{B}$ iff any two homomorphisms h and h' of $\mathcal{A}$ in $\mathcal{B}$ are such that $\widehat{h} = \widehat{h'}$.

By analogy with the case of algebras, we define the notion of subprealgebra of a prealgebra. A subprealgebra X of the prealgebra C is generated by the subset $G \subset C$ if the subprealgebra X is the smallest subprealgebra of C which contains G.

Definition 4.64. Let $\mathcal{A}_\alpha$ be a family of similar prealgebras. The free prealgebra with n generators, n being a cardinal > 0, associated to $\mathcal{A}_\alpha$, denoted by $\mathbb{F}_n(\mathcal{A}_\alpha)$, is defined as follows: 1) $\mathbb{F}_n(\mathcal{A}_\alpha)$ is generated by the generators $a_1, a_2, \ldots, a_n$; 2) any bijection of the set of generators on $\mathcal{A}$, belonging to $\mathcal{A}_\alpha$ can be extended to essentially one homomorphism of $\mathbb{F}_n(\mathcal{A}_\alpha)$ in $\mathcal{A}$.

Precisely as in the case of algebras, if a family $\mathcal{A}_\alpha$ is composed by prealgebras satisfying equivalences or implications between equivalences, then the free algebra $\mathbb{F}_n(\mathcal{A}_\alpha)$ always exists. On the other hand, two free prealgebras associated to the family $\mathcal{A}_\alpha$ are isomorphic. The free prealgebra with n generators associated to $\mathcal{A}_\alpha$ is such that any other prealgebra with n generators similar to the prealgebras of $\mathcal{A}_\alpha$ constitutes a homomorphism of $\mathbb{F}_n(\mathcal{A}_\alpha)$, if it satisfy the same equivalences or implications of equivalences as those of the prealgebras of $\mathcal{A}_\alpha$.

In order to prove the existence of the prealgebra associated to a family of prealgebras $\mathcal{A}_\alpha$, all of them verifying a set C of equivalences and implications between equivalences, we proceed as follows.

Let a_n be the family of n objects that will act as the generators of the free prealgebra whose existence we shall prove. We designate by $v_{\mathcal{A}}$ the attribution to each a_n the value $v_{\mathcal{A}}(a_n)$ that belongs to $\mathcal{A}$, a prealgebra of $\mathcal{A}_\alpha$. Such attribution is called an $\mathcal{A}$-valuation.

With the symbols of the operations of the prealgebras of $\mathcal{A}_\alpha$ and variables, we form all the terms that can be built with those symbols. Given, then, the algebra $\mathcal{A}$ of the family $\mathcal{A}_\alpha$, the $\mathcal{A}$-valuations give values to the terms, when each variable is replaced by a generator. Two such constant terms are equivalent, by definition, if the corresponding values are equivalent in $\mathcal{A}$. That equivalences are equivalences valid in $\mathcal{A}$.

Therefore, $\mathbb{F}_n(\mathcal{A}_\alpha)$ is isomorphic to a subprealgebra of the Cartesian product of the prealgebras so obtained, each prealgebra deriving of an $\mathcal{A}$ in $\mathcal{A}_\alpha$ and of an $\mathcal{A}$-valuation extended to the terms of $\mathcal{A}$. In other words, $\mathbb{F}_n(\mathcal{A}_\alpha)$ is a subdirect product of the prealgebras of $\mathcal{A}_\alpha$.

All formal languages of the common logics are free prealgebras or, in some cases, free prealgebras, as we shall see. On free algebras, see Birkhoff ([12]), and Bourbaki [13].

The nature of logic involves topological and algebraic ideas.

Though a large variety of logical notions are topological in nature (related to the weak topology determined by the theories of closed sets), logic also presents an algebraic dimension (the underlying language of most logics are free algebras, whose operations possess a profound logical signification, numerous logics are essentially lattices, etc.).

5

ALGUNS TEOREMAS DE INCOMPLETUDE

> If you wish, therefore, understand the world theoretically, so far as it can be understood, you must learn a very considerable amount of mathematics. If your interests are practical, and you only wish to manipulate the world, whether for your own profit or for that of mankind, you can, without learning much mathematics, achieve a great deal by building on the work of your predecessors. But a society which confined itself to such work would be, in a sense, parasitic on what had been already discovered.
> Bertrand Russell [93], p. 27.

Supomos que os conceitos básicos da metodologia das teorias dedutivas sejam conhecidos, ainda que por alto. Por exemplo, mencionamos os seguintes: lógica clássica de primeira ordem, lógica intuicionista, sistema formal, enumeração efetiva de um conjunto, verdade aritmética, fórmula aritmética que expressa a consistência de um sistema formal, aritmética de Peano de primeira ordem e de ordem superior, verdade na aritmética de Peano, aritmética de Heyting, consistência sintática, computador teórico ou máquina de Turing, etc. (Detalhes podem ser obtidos nas seguintes obras, entre outras: [55], [76] e [87].)

Gödel demonstrou dois metateoremas sobre a aritmética clássica que podem ser considerados como se constituindo em dois monumentos da lógica e da metodologia das disciplinas dedutivas; eles são chamados de teoremas de incompletude de Gödel e são os seguintes:

Primeiro teorema de Gödel Não existe sistema formal consistente cujos teoremas podem ser efetivamente enumerados, com ou sem repetição, no qual todas as fórmulas verdadeiras da aritmética de Peano de primeira ordem sejam demonstráveis.

Segundo teorema de Gödel Não há sistema formal, nas condições do primeiro teorema, no qual se pode demonstrar uma fórmula aritmética que efetivamente expresse a consistência do sistema.

Fundamental para as demonstrações dos teoremas precedentes é a chamada técnica de aritmetização da sintaxe, introduzida por Gödel (consultar um dos livros citados acima). Dos resultados apresentados seguem-se vários outros, tais como:

1. Existem fórmulas fechadas da aritmética de Peano de primeira ordem, suposta consistente, tais que nem elas nem suas negações são demonstráveis na referida aritmética.

2. Se a referida aritmética de Peano for consistente, então certa fórmula fechada que expressa sua consistência não é nela demonstrável.

3. Nenhuma teoria matemática usual, mais forte do que a referida aritmética e consistente, é sintaticamente completa.

4. Nas condições do teorema precedente, uma fórmula que expresse efetivamente a consistência de uma teoria matemática padrão é teorema da mesma.

5. A aritmética de Peano de primeira ordem pode ser imersa na aritmética de Heyting.

6. Se a aritmética de Heyting não for trivial (possui sentenças que não são demonstráveis), então a aritmética de Peano de primeira ordem é consistente.

   Certos resultados acima são adaptáveis ao caso das aritméticas em consideração serem de ordem superior. E algo diferente, porém significativo, se passa quando se conserva apenas a adição e se elimina a multiplicação totalmente como operação primitiva. Ademais, parte do exposto, com modificações, continua sendo válido para certas lógicas chamadas de paraconsistentes [21]. Tem=se por exemplo:

7. Se A for a aritmética baseada na lógica paracomnsistente de primeira ordem com igualdade de [31] e [21], então A não sendo trivial implica que existe uma sentença que exprime a não trivialidade de A e que, se A não for trivial, não é teorema de A (A é trivial se e só se todas as suas fórmulas forem teoremas).

6

MECÂNICA QUÂNTICA, TEORIA DE CONJUNTOS E TEOREMA DE BANACH-TARSKI

All I'm concerned with is that the theory should predict the results of measurements. Quantum theory does this very successfully.
Stephen Hawking

No que se segue, ZF e ZFC denotarão Zermelo-Fraenkel sem escolha e Zermelo-Fraenkel com escolha, ambos com um conjunto (infinito) enumerável de elementos irredutíveis ou, simplesmente, de irredutíveis (também chamados de Urelemente de átomos ou de indiscerníveis). Supomos ZFC consistente (ver [53] e [54], bem como as correspondentes referências aos temas deste artigo).

Tem-se:

Teorema 6.1. *Há um modelo M de ZF no qual todos os conjuntos são invariantes por permutações de irredutíveis.*

A demonstração é feita por indução transfinita (e metateórica).

Teorema 6.2. *Em M, modelo de ZF, existe o conjunto F de todos os pares (não ordenados) de irredutíveis distintos.*

Teorema 6.3. *Existe uma extensão de M, M', modelo de ZFC, no qual há um conjunto escolha de F, F', e F' não é invariante por permutações de irredutíveis.*

Corolário 6.4. *F' existe em ZFC mas não em ZF.*

Teorema 6.5 (Fraenkel)**.** *O axioma da escolha não é teorema de ZF.*

Teorema 6.6. *A negação do axioma da escolha não é teorema de ZF. Logo, o axioma da escolha é independente dos demais axiomas de ZFC.*

Nota 1 No caso em que ZFC (e portanto ZF) não contiver irredutíveis, o referido axioma ainda é independente dos demais axiomas de ZFC, como demonstraran Gödel e Cohen, recorrendo a métodos diferentes dos de Fraenkel.

O axioma da escolha é extremamente importante para o desenvolvimento matemático da mecânica quântica. Constata-se isso, por exemplo, pelo seguinte

Teorema 6.7. *Em certos modelos de ZF, há espaços vetoriais, em particular espaços de Hilbert, sem bases. Ademais, existem espaços vetoriais com bases de cardinais diferentes.*

Nota 2 Os irredutíveis em ZF possuem características dignas de destaque; por exemplo, não existe nenhuma propriedade conjuntista em ZF que possa distinguir dois indistingíveis quaisquer entre eles ou, em outros termos, eles são indiscerníveis na acepção usual da palavra. Em ZFC, em que o conjunto dos irredutíveis tem cardinal não enumerável, não se pode distinguir dois irredutíveis quaisquer. A analogia entre irredutíveis em ZFC (ou em ZF) e partículas quânticas é óbvia. Outro aspecto interessante: podemos bem ordenar, em princípio, os números reais, porém, na linguagem de ZFC, não há nenhuma maneira de se descrever uma tal boa ordem.

É verdadeira a seguinte proposição:

Teorema 6.8. *As relações matemáticas em mecânica quântica, baseada em ZFC, são invariantes por permutações de partículas da mesma espécie.*

Nota 3 O teorema anterior está claramente conectado com a mudança ou não de sinal na função de onda, conduzindo à distinção entre fermions e bosons.

Nota 4 Pode-se demonstrar (teorema de Vitali) que, na reta euclidiana, há subconjuntos não mensuráveis segundo Lebesgue. Para isso, necessita-se do axioma da escolha.

Porém, existe a chamada matemática de Solovay, na qual todos os subconjuntos da reta são mensuráveis (no sentido de Lebesgue). Esta matemática tem por base ZF, com ou sem indiscerníveis, com adição do axioma das escolhas dependents, que é um caso particular do axioma da escolha. Nesta matemática, há espaços de Hilbert sem bases e o axioma geral da escolha não é verdadeiro, entre outros resultados significativos.

Nota 5 Para se provar que a união enumerável de conjuntos enumeráveis é enumerável, emprega-se o axioma da escolha de modo essencial. O mesmo ocorre para se demonstrar que as definições usuais de limite de uma função (por sequências ou pela técnica dos epsilons e dos deltas) são equivalentes e, também, na demonstração da equivalência de definições de conjunto finito (sem o referido axioma, haveria uma infinidade de tipos de conjunto finito).

1. O teorema de Banach e Tarski

Um dos mais surpreendentes teoremas da matemática é o de Banach-Tarski. Alguns autores o denominam de *Paradoxo de Banach-Tarski*. Existem diversas versões deste teorema. Uma dela é a seguinte:

Teorema 6.9 (Banach-Tarski)**.** *Qualquer esfera (sólida) do espaço euclidiano tridimensional pode ser decomposta em um número finito de partes disjuntas que podem ser justapostas (sem modificá-las) de modo a se obter duas cópias idênticas da esfera original.*

Para detalhes sobre o teorema precedente, consulte-se [100] e as obras neles citadas. O teorema anterior mais parece um tremendo paradoxo, que jamais poderá ter um significado físico. Muitas vezes é citado para evidenciar como a matemática se afasta da realidade física. De fato, ele parece comprovar que há um certo vácuo entre nossa intuição e os conceitos aos quais recorremos para dominar e entender a natureza . . .

2. Conclusões

1. É claro que a mecânica quântica não resolve todos os problemas empíricos a ela relacionados. Por outro lado, matematicamente se pode substituir, em uma experiência, dada partícula por outra da mesma espécie, sem se infringir os princípios gerais da mecânica em apreço. Assim, quando Dehmelt prendeu uma partícula em "gaiola" apropriada, isolando a partícula *Priscilla*, pode batizá-la e estudar várias de suas propriedades. Obviamente. Priscilla poderia ser qualquer outra partícula de sua espécie. Mas a verdade foi que Dehmelt isolou e batizou Priscilla: ele a identificou em sentido lógico, diferenciando-a de qualquer outra, a ela semelhante ou não ([3]).

2. O teorema de Fraenkel exibe o caráter abstrato da teoria de conjuntos, teoria que serve de alicerce para a mecânica quântica, bem como ilumina o modo pelo qual essa mecânica se afasta de nossa experiência quotidiana: ela é, sobretudo, uma construção nossa que, na melhor das hipóteses, apenas reflete aspectos do "real".

3. Sem o axioma da escolha, não há uma teoria como a usual dos espaços de Hilbert, central para a mecânica quântica padrão, fato que von Neumann mostrou. É digno de nota que tais teorias devam tanto a algo tão abstrato. . .

4. Finalmente, o teorema de Banach-Tarski sublinha como se tem de recorrer a construções matemáticas tão afastadas da experiência comum para se dar conta de fenômenos naturais. Sistematiza-se o concreto via técnicas e procedimentos abstratos . . .

Parte II

Fundamentos da Ciência

7

REMARKS ON QUANTUM MECHANICS (SUMMARY)

Ce qui est simple est toujour faux. Ce qui ne l'est pas est inutilisable.
Paul Valéry

In this note we make some comments on the foundations of non-relativistic quantum theory. We discuss questions related to the concept of identity, the meaning of experience and aspects of space- time.

I

In what follows, ZFC′ will denote the system of Zermelo-Fraenkel with a finite or infinitely denumerable set of Urelemente, formalized in the first-order predicate calculus without identity. ZFC will be Zermelo-Fraenkel systematized in the first-order logic with identity. ZFC′ and ZFC will include the axiom of choice. Under these conditions, it is not difficult to prove the following statements:

Theorem 7.1. *ZFC′ is consistent if, and only if, ZFC is consistent.*

Theorem 7.2. *ZFC is a conservative extension of ZFC′.*

Theorem 7.3. *Let T' be a theory based on the first-order predicate calculus without identity, but with Urelemente, and T an extension of T' by de addition of identity (and its postulates). Then, if T' is consistent, so is T.*

Corollary 7.4. *If the theory of quasi-sets, QS, is consistent, then so is QS', the theory QS plus identity.*

Theorem 7.5. *If we adjoin to QS' the axiom that indiscernible Urelemente are identical, we get ZFC. QS', so extended, and ZFC are equiconsistent. (Similarly, when the set of Urelemente is a finite family of disjoint finite sets of indiscernible elements.)*

Remark In the extant empirical sciences, therefore, such as physics, given a consistent formulation of a theory based on ZFC, there are no means to distinguish between identity and congruence (a binary relation having the formal properties of identity: reflexivity, symmetry, transitivity and substitution). This fact may be called the indetermination of identity and has to be taken into account in any systematic treatment of the sciences, QM in particular. This indetermination is a logical problem, but, from the ontological perspective, numerous ways are open to overcome this difficulty.

II

The part of quantum mechanics constituted by the techniques to test its empirical consequences, involves probability, statistics, computing and the planning of experiment. In this case, the employment of classical mathematics and its techniques is essential and cannot be eliminated, at least today. Therefore (QM abbreviates Quantum Mechanics):

Theorem 7.6. *QM is composed by two parts: the first, theoretical, which may be systematized (the empirical content being preserved) by the means of several alternative logics, and the second, involving experimentation, based on classical logic and mathematics.*

Theorem 7.7. *If a non classical logic, L, is employed to systematize quantum mechanics, since its empirical part is classical, then a central task is to investigate the relations between L and classical logic.*

Theorem 7.8. *In principle, the theoretical part of QM may only be seen as a method to help us to get new experimental results. (The black box view.)*

Theorem 7.9. *The mathematical relations valid in QM, based on ZFC, are invariant under permutations of particles of the same species.*

Theorem 7.10. *The preceding proposition remains true in QS if the particles, supposed to be identical, are only indiscernible.*

III

One of the most surprising results in the mathematics of the three-dimensional Euclidean space is the theorem of Banach-Tarski, also called paradox of Banach-Tarski. Among its possible formulations, there is the following:

Theorem of Banach-Tarski.- Any solid sphere of the three-dimensional Euclidean space can be decomposed into a finite number of disjoined parts that can be juxtaposed in such a manner that we obtain two identical copies of the original sphere. (The axiom of choice is utilized in the proof of the theorem.)

The above theorem shows how the underlying space-time of QM is an idealized structure. (Banach-Tarski's theorem is an extension of Hausdorff's theorem which refers to the spherical surface.) We have:

Theorem 7.11. *The Euclidean space-time of QM does not express the 'real' space-time connections of extant particles. (This is valid, of course, in connection with any relevant physical theory; however, in QM it obviously conveys a false image of the physical situation, which causes various theoretical difficulties.)*

IV

As it is well known, axiomatic theories have infinitely many possible interpretations, but normally only one among them is considered as the "true" one. In QM this remark also applies; but in this discipline the state of affairs is more complex:

Theorem 7.12. *QM has, in principle, various alternative interpretations, all of them being empirically equivalent. So, at least today, there are various quantum mechanics.*

Theorem 7.13. *One of the difficulties is the size of particles. If they behave like material points, there are some obstacles; otherwise, there are also problems of different nature. (There are several analogous problems.)*

Conclusions

1. There exist various empirically equivalent, though theoretically distinct, formulations of quantum mechanics, even based on non classical logics. In certain sense, quantum mechanics is the collection of these incompatible formulations.

2. Classical logic and mathematics are essential to the complete systematization of such mechanics, with reference to its empirical applications.

3. Maybe, the theoretical difficulties presented by quantum mechanics are, in general terms, a consequence of the fact that it is only a first approximation to a more sophisticate and more complete theoretical construction.

4. Clearly, the central portions of its various formulations may be employed as bases for several alternative ontological developments.

8

GÖDEL AND THE FOUNDATIONS OF PHYSIC

OUR PRINCIPAL aim in this paper is to analyze some aspects of Gödel's incompleteness theorems in connection with physics. It deserves to be mentioned that in fact Gödel seems never to have written any commentary on this topic. On the other hand, only a few experts did, in earnest, investigated such subject, deserving mention, among them, Stanley Jaki and Stephen Hawking.

We will not present here a rigorous and full treatment of the theme, but proceed informally, giving references for the reader, in the case that he is interested in an ample and rigorous treatment of the matter.

Good expositions of Gödel's incompleteness theorems may be found in Mendelson [87] and Kleene [76]. A more elementary exposition is that of Franzén [55]. Hawking [63] and Jaki [72] are very good expositive pieces, both elementary.

All syntactic concepts to which we make reference in this paper are arithmetically definable, i.e., can be expressed in first order Peano arithmetic (Franzén [55]).

1. Gödel's incompleteness theorems

Normally, the mathematical theories are presented in an informal, intuitive way. Nonetheless, we are able to transform this subject in axiomatic theories and, at least in principle, into formalized pieces, that is, in a certain sense, into plays with symbols (see [55], [73] and [76]). This can be done with Peano arithmetic, set theory and other mathematical theories, including those of physics supposed rigorously systematized. In other words: the theories of mathematics and physics are in principle reducible to symbolic games, that is formalizable.

Giving a formalized theory, the next possible step is to transform the initial theory in part of Peano arithmetic, that is, to arithmetize it: to transform the initial theory in a portion of arithmetic. So, it becomes an arithmetical theory, involving only natural numbers, properties of numbers and relations involving numbers.

Summarizing Gödel showed that all extant mathematics can be immersed in Peano arithmetic (of first order): the whole of mathematics becomes only arithmetic.

This way, all significant and recursive aspects of arithmetic, from the syntactical level, are expressible in the language of first order Peano arithmetic. The concept of recursive enumerability is of basic relevance for the proof of Gödel's theorems; in fact, it is possible to prove that the theorems of a formalized mathematical theory constitute a set that is recursively enumerated, with or without repetitions, by a general recursive function of the set of natural numbers in the same set. In technical words, that the class of formal theorems is recursively enumerable.

As a consequence, it is possible to prove Gödel's first theorem:

If a theory containing Peano arithmetic of first order is consistent, then it is syntactically incomplete, that is, there are arithmetical sentences such that neither of them nor their negations are provable in that theory.

Gödel's second theorem is the following:

If a theory supposed to contain Peano arithmetic is consistent, then it is unable to prove its own consistency (that is expressible by a recursive arithmetical first order formula, as shown by Gödel).

2. Incompleteness in physics

Physics is an area of science in which the preceding general considerations may be applied. The Sixty Problem belonging to the list of Hilbert [67] is the following:

To develop an axiomatic treatment for all physics. Clearly, such axiom system should be strong enough to contain at least first order Peano arithmetic; in addition, it should be consistent, i.e., it does not contain contradictions (in classical, standard logic, a single contradiction would entail that all sentences of the underlying language of the theory would be provable, case in which the theory is called trivial). But taking into account Gödel's first theorem, Hilbert's program would be impossible, since mathematics would be trivial.

Gödel's second incompleteness theorem constitutes a corollary to the proof of the first.

This means that under certain clear conditions, the consistency of a consistent physical (or mathematical) theory can not be proved inside this theory.

Gödel's first theorem shows that some today (hypothetical) theories, such as the Theory of Everything (TOE) and a complete and consistent theory of strings (and also the theory M) are impossible. Analogously, extended theories similar to a complete theory of the Standard Model and the Grand Unified Theory (GUT), supposed consistent, are also impossible.

Moreover, even the consistency of those theories can not be proved inside themselves if they are consistent.

It seems that Jaki [72] and Hawking [63] were two of the first authors to call seriously the attention of the experts for the applications of Gödel's first theorem to physics.

Gödel's theorems, as presented above, are true of physics based on classical logic. But they remain true in the case of certain non-classical logics, if these logics are employed as the foundations of the corresponding physics. Thus, the metatheorems are valid in the cases of some intuitionist version of mathematics, as well as in formulations of some paraconsistent mathematics and their logics.

Obviously, the theorems of Gödel remain true for other sciences, like chemistry and astronomy. An important point is that they are also applicable to various fields of technology.

Similarly, we could submit Tarski's theorem on truth to an analogous analysis as that of Gödel's theorem (see [55] on Tarski's theorem).

We end this section reproducing some considerations made by Jaki [73]:

> Herein lies the ultimate bearing of Gödel's theorem on physics. It does not mean at all the end of physics. It means only the death knell on endeavours that aim at a final theory according to which the physical world is what it is and cannot be anything else. Gödel's theorem does not mean that physicists cannot come up with a theory of everything or TOE in short. They can hit upon a theory which at the moment of its formulation would give an explanation of all known phenomena. In terms of Gödel's theorem such a theory cannot be taken for something which is necessarily true. Apart from Gödel's theorem, such a theory cannot be a guarantee that in the future nothing essentially new would be discovered in the physical universe which would then demand another final theory and so on. Regress to infinity is no answer to a question that keeps generating itself with each answer.

3. Concluding remarks

Gödel's theorems and analogous results show that the scientific activity is an open and unfinishable task. Maybe, a kind of Sisyphus job.

Moreover, since we effectively don't know how are reality, our circumstance or our universe, perhaps physics and the other sciences will profoundly change in the future. In the long run, science may change drastically. As Hawking puts it ([64]):

> I don't demand that a theory correspond to reality because I don't know what it is. Reality is not a quality you can test with litmus paper. All I'm concerned with is that the theory should predict the results of measurements. Quantum theory does this very successfully.

Another passage of Jaki is pertinent here [73]:

> Gödel's theorem means, among other things, that physicists who aim at reading God's mind will not succeed, because they cannot read their own minds in the first place — Gödel's theorem remains a serious assurance to all physicists that their minds will forever be challenged by ever fresh

> problems. With recourse to logic they would also know what to think of efforts to derive the very specific constants of physics form non-specific considerations. Insofar as mathematics works with numbers it will remain steeped in specifics all of which raise the question: Why such and not something else? It is that question which keeps the mind awake, or rather is raised by minds not prone to slumber.

In synthesis, the scientific endeavour constitutes an endless activity. Gödel's theorems and other incompleteness results aren't the final act of scientific activity, but they suggest that it is an eternal task.

4. Complementary note

There are other ways to find limitations in the extant, usual mathematics, as well as in the sciences in general (confer with [43]). Adapting the conclusion of [43], to our present situation, we could assert: "Provability in math is rare, a rare event".

So, there is a major question to be dealt with here: why does it seem to us that all major mathematical questions can be settled by our main provability techniques?

There is no easy answer to that question, but we are slowly uncovering a wilderness of mathematical facts that stay out of the reach of our main mathematical tools.

As if mathematics would remain inaccessible from itself. For mathematics can be described as Dante said, "as selva selvaggia, as a savage, nearly impassable, wilderness."

The above passage, we believe, is true of all natural sciences.

Gödel presented his incompleteness results only in connection with mathematics. However, in 1996, in his book [72], Jaki remarked that the same is true with reference to physics. Talking about a meeting in which various very known physicists participate including him, he wrote in [73] that:

> After the speech [of Gell-Mann] it was first the turn of the other panelists to comment. When my turn came I reminded Gell-Mann that even if he had formulated such a final theory he could never be sure that it was really final. He shouted back rather angrily. "Why not?" "Because of Gödel's theorem" I replied. "Whose theorem?", he asked again. I said

> again "Gödel's theorem". Then I had to spell out Gödel's name which Gell-Mann apparently had not heard before.

It is obvious that Gödel's theorem remains true for almost all scientific theories that are not too weak.

9

ON THE FOUNDATIONS OF QUANTUM MECHANICS

IN THIS CHAPTER, we consider the nonrelativistic and nondeterministic quantum mechanics (abbreviated by QM) in the version of Schrödinger ([96]; in this reference there are summaries of other versions of QM including the one of de Broglie-Bohm which is deterministic). Our objective is to present some known metatheoretical results of particular interest for our principal aim: to discuss certain fundamental aspects of QM and to delineate some philosophical consequences of our discussion.

1. Preliminary remarks

The underlying space-time of QM is the classical one of Galileo-Newton. From the mathematical point of view QM constitutes the study of a kind of complex (in particular real) Hilbert space supposed to be separable.

A hidden-variable theory is one in which the description of a quantum system by the means of states described by QM is not a complete description. New variables, beyond those of standard QM contribute to govern the quantum processes.

Local hidden-variables are ruled out by Bell's theorem stated in the next section. On the other hand, a satisfactory non- local hidden-variable theory was worked out by de Broglie, Bohm and others ([96]).

The various standard versions of QM (see, for example, [59] and [96]) are empirically equivalent. However, some topics can be treated more conveniently in one version than in another.

2. Bell's theorem

There are various so-called formulations of Bell's theorem, although they are not strictly equivalent. For instance, the following ([9]):

1. If a hidden-variable theory is local, it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local.

2. No physical theory of local hidden-variables can ever reproduce all predictions of quantum mechanics.

3. In a non-local hidden-variables theory, there are signals that propagate instantaneously and the theory is not Lorentz invariant.

Bell's theorem implies that there exist no hidden-variables in QM, what is a fundamental aspect of this theory. But, on the other hand, we have:

Theorem 9.1. *Usual quantum field theories, which are local, are not compatible with QM unless some ad hoc adaptations are made.*

3. Kochen-Specker theorem

The theorem of Kochen-Specker is essentially connected with the one of Bell; it is the following ([81], [85] and [**?**]):

Theorem 9.2 (Kochen-Specker)**.** *It is impossible to attribute values, simultaneously, to all quantum quantities of a quantum system, under the supposition that the relations existing among them, imposed by QM, remain valid. (The dimension of the corresponding Hilbert space has to be equal to or greater than 3.)*

Among the suppositions behind the proof of the Kochen-Specker theorem, there is the following: The results observed in measurement are dependent upon what other measurements are being made; in other words, the result of a measurement of an observable is dependent on which other commutating observables are being measured. (Quantum contextuality means that the result of a measurement of a quantum observable is dependent of which other commuting observables are being regarded.)

We also have:

Hidden variables are correlated with the system that is being measured and not with the process of measurement. A non-contextual hidden-variable theory cannot predict the usual results of QM.

As a consequence of the preceding propositions, quantum mechanics is "contextual". In it there are no properties or relations in the usual meaning of the word (we have to take into account the context).

This way there exists in QM a logical difficulty: the properties and relations of a quantum system are context-dependent. Therefore we arrived at a problem: the logical laws of classical logic, in particular the principles of first-order predicate calculus, with or without equality, are not valid for the quantum systems. However, classical logic is usually employed as a foundation for QM. How is this possible? (Sometimes, only parts of classical logic or some informal modifications of parts of such logic are taken into account. But usually there are no detailed indications of how our problem could be solved by this kind of procedure.)

We insist on the fact that the Kochen-Specker theorem shows that the classical, metaphysical, identity is not valid in QM. Effectively, in order to be valid it would be necessary that each observable of a quantum system should have a well- defined value in any instant of time, what is false according to the theorem.

4. The theorem of Conway-Kochen

The proposition to be taken into account in this section, also called the Free Will Theorem, is the following ([16]):

Theorem 9.3 (Conway-Kochen)**.** *Under certain conditions, satisfied by QM, if two physicists are free to make choices on what to be measured, then the* *results of the measurements can not be determined by what did happen before the experiments (quantum indeterminism).*

For us, the relevance of the preceding proposition is, above all, that it implies the fact that QM constitutes, essentially, an intrinsically probabilistic discipline: there is no simple deterministic extension of it. (Loosely speaking, a simple extension of a theory is another theory, stronger than the first formulated in the language of the initial theory: there are no new additional concepts.) In connection with QM one has to be careful when it is employed in combination with other disciplines, specially deterministic ones (such as, for instance, relativity). The interconnections between macroscopic determinism and QM are delicate and difficult to be explained.

5. Gleason theorem

In this section, we deal with Gleason's theorem (see [57]), which is the following proposition:

Theorem 9.4 (Gleason)**.** *Gleason theorem: Let H be an Hilbert space, real or complex, of dimension greater than or equal to 3, and u a measure on the closed subspaces of H. Then. there exists a positive and self-adjoint operator T, of the trace-class, such that, for all closed subspaces L of H, one has: u(L)= tr (TPL), where PL is the orthogonal projection on L ([57]).*

The above proposition implies Born's rule to obtain the probability of a measure, the non existence of local hidden-variables in QM and the contextual nature of this discipline. In a few words, Gleason's theorem describes the mathematical heart of QM and seems to suggest a new basis for a general probabilistic approach to the world. (According to Pitowski, QM constitutes a kind of probability theory ([90]).)

6. Other metatheorems on QM

Let us suppose that S is a quantum system associated to a Hilbert space H. It is possible to decompose any vector of H according to infinite bases and all the identities that express such decompositions can not be simultaneously true by the theorem of Kochen-Specker (we have to limit the context so to say).

In consequence of that fact, we have:

Theorem 9.5. *Theorem 6.1: QM is a probabilistic discipline that cannot be extended to a consistent and deterministic one, unless its language is enlarged.*

Theorem 9.6. *Theorem 6.2: In QM, there are algebraic bases which are not physically meaningful.*

Remark: "Physical object" or "physical system" is a notion that was created and developed through the history of physics; it is not a true concept that has been discovered in a Platonic heaven; and neither is a concept that must necessarily constrain our present and future physical theories. In particular, "it might not constrain QM." ([44])

Theorem 9.7. *Let S be a quantum system associated, by QM, to the Hilbert space H in some interval of time I. Under these conditions, there are during I, mathematically true propositions referring to H which are not physically true of S.*

The central results of this paper constitute metatheoretical ones, that help us to make an idea of the meaning of the physical theories and, as a consequence, of physics. We may say that the logical significance of physics is profoundly connected with its central metatheoretical results. Those presented above are related to the foundations of quantum mechanics. But more or less similar ones are appearing in all areas of physics (for instance, the Kolmogorov-Arnold-Moser theorem in celestial mechanics ([6])). Today the progress in the metatheoretical foundations of science in general is in a continuous state of development, such as the one of physics in particular.

7. An informal digression

In informal quantum mechanics there are some difficulties caused by informality. An example is the following: In order to deal with Schrödinger equation, we assume that quantum particles are essentially points. If they are not, we could, as it is normal, to consider, instead of particles, their centers of gravity, such it is done in classical mechanics. However, this and other categories of procedures give rise to insurmountable difficulties (form of the particles, their trajectories, their volumes, etc).

On the other hand, if, for example, an electron and a positron were points, there would be in certain cases of sufficient approximation owing to Coulomb's law, an infinite attraction. So we have:

Theorem 9.8. *Informal QM is inconsistent.*

The preceding statement shows that certain amount of rigor is relevant, even if we are not trying to develop an entirely formal QM.

8. Kinds of physical theories

At the beginning of this section we quote Manin ([85], p. 78):

> The development of the foundations of physics in the twentieth century has taught us a serious lesson. Creating and understanding these foundations turned out to have little to do with the epistemological abstractions that were of such importance to the twentieth-century critics of the foundations of mathematics: finiteness, consistency, constructibility , and in general, the Cartesian notion of intuitive clarity. Instead, completely unforeseen principles moved into the spotlight: complememtarity, and a non classical, probabilistic truth function. The electron is infinite, capricious, and free, and does not at all share our love for algorithms.

There are three kinds of physical theories:

1. Those which possess definite mathematical languages in which they can be formalized. For example, Euclidean geometry and classical point mechanics, both envisaged as natural sciences. They have a strict axiomatic formulations, as well as informal meanings.

2. Theories which, in certain acceptation, do not have a proper language using instead the language of a mathematical discipline conveniently interpreted. This happens with QM: its language is that of the theory of Hilbert spaces employed from the perspective of physics. This situation is similar to that of quantum field theory as in the standard Model.

3. In general, a physical theory has a place in between the preceding two cases such that it happens with hydrodynamics and heat theory.

A physical theory in particular QM, must contain connections with experience This relationship is build with the help of probability, numerical analysis. statistics the theory of errors, the planning of experience and other procedures. We finish this part of our paper quoting again Manin ([85], p. 80):

> We have s somewhat unusual situation in that quantum mechanics does not really have its own language. More precisely, to describe a physical system S such as a 'free electron', 'atom of helium in a magnetic field', etc., quantum mechanics uses a certain fragment of functional analysis, 'oriented on describing S'. [Clearly the language employed is that of the theory of Hilbert spaces.]

9. Final remarks

According to Bell's theorem, no local hidden variables are compatible with QM. In other words, QM is nonlocal and nondeterministic. So, QM is incompatible with strict field theories. In a few words, the effective logic of physics has to be paraconsistent, although locally classical (see [38] and [85]).

The metatheorem of Kochen-Specker does confirm that the underlying logic of QM isn't classical, but locally classical.

The result of Conway-Kochen clearly proves that determinism is out of the range of QM. Gleason's metatheorem makes it evident that most of the above traits are inherent in the structure of QM, confirming that the effective underlying logic of this branch of knowledge is non classical.

In addition, by section 7 of this paper, we conclude that if we are not careful, we will arrive at contradictions. A certain amount of rigor appears as basic for QM; otherwise, we shall find enormous conceptual hindrances.

The previous discussion concerning a possible and general classification of physical theories indicated that ours does constitute a reasonable one.

Since QM isn't Lorentz invariant, it is, strictly speaking, incompatible with special relativity and common field theories (see, for instance, [60] and [83]). There exists, however, a certain agreement among all those theories:

they are locally and empirically harmonious. This means in fact that the underlying, working logic of quantum theory is paraconsistent in a non local sense ([38] and [44]), topic that we shall treat in a following paper.

A large portion of the present work can be adapted to the case of quantum field theory, as we shall explain in the future.

10

A LÓGICA DA IDENTIDADE E A MECÂNICA QUÂNTICA

OS SEGUINTES princípios acham-se explícita ou implicitamente na lógica e na metafísica de Aristóteles:

1. Princípio da existência (PE): Objetos do conhecimento existem ;
2. Princípio da identidade (PI): Qualquer entidade é idêntica a si mesma;
3. Princípio da não contradição (PNC): Nenhuma entidade possui propriedades contraditórias;
4. Princípio do terceiro excluído (PTE): Uma entidade possui ou não possui uma dada propriedade.

Na lógica atual, tais princípios se apresentam com formulações precisas e a identidade está sujeita à lei de Leibniz além de outras características. Denominaremos de identidade metafísica (ou aristotélica) uma relação de identidade assim concebida, envolvendo os princípios intuitivos precedentes.

Tem-se então:

Em primeiro lugar, o teorema de Kochen-Specker mostra que a identidade metafísica não é válida em mecânica quântica (MQ não relativista).

De fato, para a identidade metafísica valer seria preciso que cada observável de um sistema quântico tivesse um valor bem definido em qualquer instante do tempo o que não ocorre pelo teorema de Kochen-Specker: a MQ é *contextual*.

Em segundo, a lei de Leibniz não tem sentido para a identidade em MQ. Como esta é contextual, a lei em consideração não pode ser aplicada em geral.

Tendo-se em mente a exposição precedente, verifica-se que a teoria clássica de conjuntos afigura-se incompatível com o teorema de Kochen-Specker. Ao que tudo parece indicar é que a lógica da MQ tem que ser "local".

A argumentação anterior tem por base o artigo: da Costa, N.C.A e de Ronde, C., Critical remarks on the applicability of metaphysical identity in Quantum Mechanics [44].

Observe-se que uma teoria da identidade fora do espaço-tempo não pode ser nem correta e nem completa.

Tendo-se em mente o teorema de Kochen-Specker, a noção de identidade passa a ser, portanto, algo "local", estritamente não clássico. Lógicas que derrogam leis da identidade clássica (metafísica e aristotélica) poderiam ser chamadas de lógicas com identidades parciais. Tais lógicas são lógicas não clássicas. No tocante à identidade em MQ, o melhor, talvez, fosse denominá-la apenas de congruência. Por outro lado, pode-se, repitamos, sustentar que uma teoria da identidade desconectada do espaço e do tempo não pode ser nem correta nem completa.

Neste ponto chega-se a um dificuldade conceitual: ao que tudo indica, a lógica da MQ tem que ser contextual mas, não obstante, aplica-se a lógica tradicional de modo universal; pelo menos é isso que acontece nas exposições comuns da MQ.

Há outra questão significativa: o reticulado das proposições quânticas não é uma álgebra de Boole. Por conseguinte, corresponde a uma lógica elementar (cálculo proposicional e cálculo de predicados, com ou sem identidade) não clássica; todavia, para se mostrar este fato, recorre-se, comumente à lógica clássica. Tem-se aqui uma "combinação" de lógicas? Como se pode explicar tudo isso?

Note-se que, na contraparte experimental da MQ, a lógica e a matemática utilizadas são clássicas. Efetivamente, não se dispõe, até hoje, por exemplo, nem de uma teoria dos erros de aproximação e nem de uma estatística que não sejam clássicas.

Uma das maneiras de se superar as dificuldades dos três parágrafos anteriores consiste em se recorrer às álgebras parciais, como sugere Manin [85].

A lógica da MQ está intimamente ligada à noção de probabilidade. E para se entender como isto se dá, devemos nos referir ao teorema de Gleason.

A probabilidade associada a uma proposição quântica P , à qual corresponde o projetor $\hat{P}$, na matriz de densidade $\hat{r}$, é dada pela formula

$$Prob(P,r) = tr(\hat{r}.\hat{P}).$$

O teorema de Gleason, um dos mais importantes da QM, é o seguinte:

Se a dimensão do espaço de Hilbert complexo (e de base no máximo enumerável) for maior do que 2 , a única maneira de se definir uma medida de probabilidade no reticulado das proposições quânticas é por meio da formula acima, dada a matriz de densidade r.

O teorema impõe severas limitações a mudanças no formalismo quântico, por exemplo para a introdução de variáveis ocultas. Ademais, põe em relevo o papel das matrizes de densidade. Como os operadores da QM são combinações de projeções, o teorema também se aplica a operadores em geral. (Estes são combinações lineares de projeções.)

Do teorema de Gleason pode-se deduzir a fórmula de Born para o cálculo de probabilidades quânticas. Ele também deixa claro que as medidas de probabilidade da QM são determinadas pelo formalismo hilbertiano da mesma; daí seu caráter basicamente probabilístico. Demonstra-se, a artir do teorema de Gleason, que não há medidas bivalentes em QM, o que quer dizer a inexistência de variáveis ocultas na mecânica em apreço.

Notas

1. A MQ pode ser vista como um novo cálculo de probabilidades, que se tem tentado aplicar, por exemplo, em economia.

2. Não se sabe se podemos edificar uma MQ, em vez de partirmos do corpo complexo, em particular do real, uma outra sobre os quartênios ou sobre os números *p*-ádicos.

3. As propriedades espaço-temporais podem "separar" partículas quânticas. Quando isto é possível, há identidade; em caso contrario, parece não haver identidade bem definida.

4. Objetos físicos microscópicos, em dados fenômenos, podem não apresentar identidade, possuir um *quid*, embora em outros isso seja possível. Assim, partículas separadas espaço-temporalmente e de mesma espécie possuem identidade devido à separabilidade espaço-temporal. Por outro lado, as distribuições de Bose-Einstein e de Fermi-Dirac evidenciam que, nas situações correspondents, existem objetos quânticos que por assim dizer "perderam" a identidade.

5. Dado que a QM é uma disciplina "aproximada", como pode ser base de uma ontologia não sendo "verdadeira"? Ou será apenas uma teoria que não reflete muito bem o real?

6. Em [95], [104] e [69] há importantes referências ao problema da identidade de partículas, tratados de um prisma atual.

11

COHOMOLOGY AND QUANTUM MECHANICS

(Written with Joseph Kouneiher.)

We make some remarks on the mathematical foundations of quantum mechanics. Our main objective is to insist on the relevance of cohomological methods for the treatment of such mechanics. Our present work will be the starting point of a series of papers on a new mathematical foundation for quantum mechanics and its philosophy.

1. Two kinds of quantum mechanics. There are two usual approaches to quantum mechanics: the non relativistic and the relativistic. Here we don't make this distinction and for quantum mechanics we mean the general study of quantum phenomena.

2. The meaning of cohomology. Cohomology is a form of associating algebraic invariants to mathematical structures; for example, abelian groups to a topological space. Cohomology is a kind of dual operation to homology. Both are methods of construction of invariants for mathematical structures that belong to algebra or topology (more in general, of any kind of structures at all, when this is possible and convenient). Here we shall outline how cohomology can be relevant in the domain of quantum theory. (Examples of cohomology: cohomology of a closed orientable manifold (Poincaré structure

that he used in his proof of the duality theorem), and sheaf cohomology that is a kind of generalization of the so called singular cohomology.)

3. The Aharonov-Bohm effect. This effect is a quantum phenomenon in which an electrically charged particle is affected by an electromagnetic potential in spite of being confined to a region in which both the electric and the magnetic fields are null. (This fact is a consequence of the coupling of the electromagnetic potential with the complex phase of the charged particle.) The effect under consideration can be detected by the means of interference experiments. An example occurs when the wave function of a charged particle, that pass near a solenoid, experiences a change of phase as a result of the action of the magnetic field inside the solenoid; all these things happen when the magnetic field is null in the region in which the particle moves, although its wave function is also almost null in the interior of the solenoid. Leaving aside the common explanations (for instance, that we can only measure absolute values of the wave function), with the help of cohomology methods plus some other mathematical tools, we are able to explain the effect (see [46] and [2]).

4. Conclusion. Our explanation of the effect makes appeal to cohomology with some more mathematics, as we already noted (see [46]). But in future papers we will show that with topological methods and a little more mathematics, it is possible to develop a theory that unites all aspects of quantum mechanics, non- relativistic as well as relativistic. In particular, we obtain an explanation of the Heisenberg complementarity relations, Aharonov-Bohm effect, superposition and entanglement (see [45]).

Note : In [46] and [45], we show that the existence of two fundamental objects of quantum mechanics (QM) is derived from the symmetry of space-time: the existence of particles and that of fields. The essential nature of QM is coded in the algebra of Poincaré symmetry group. If we consider only Galileo group, we miss the central part of QM and are unable to understand the real meaning of this science. In fact, we need the central extension of Poincaré group (i.e., the part which commutes with Poincaré algebra), which is linked to cohomology. Otherwise, we would consider that QM is in essence random, because the space-time correlation is left out, correlation which is responsible for its random aspect. We have to take into account all Poincaré group plus its central extension if we want to understand QM; this way we are able to perceive that the theory is at the same time global and local: this gives us entanglement and shows that cohomology is the natural mathematics for QM. Poincaré was the first to understand the situation when he introduced homology and used what today we call cohomology. There exists an analogy between Poincaré ideas and Cauchy residual theorem, where the residual integral relates the local and the global, and so it is possible to integrate. Particles or waves are only two ways to look the reality from the space side or from the time side. When we face the situation from both sides, them the two collapse and we find the quantum objects properly speaking.

Notes: 1) Null curvature does not imply that parallel transport is trivial in simple connected regions. 2) Connection is more fundamental than curvature (at least in some cases). 3) A possible "philosophical application": the potential, in electromagnetic theory, seems to be "real" and not a mathematical device.

12

SOBRE OS TEOREMAS DE BELL E DE KOCHEN-SPECKER

> Either the human mathematical mind cannot be captured by an algorithm or there are absolutely undecidable problems.
> Kurt Gödel (l951)

UMA DAS SUPOSIÇÕES de qualquer teoria da física é a de que existem regularidades de natureza física, as quais indicam que há objetos independentes de nós. Em particular, o mesmo ocorre com a mecânica quântica não relativista (MQ), na qual tratamos de objetos quânticos. Implicitamente se supõe, como no caso da física tradicional, clássica, que para se tratar de qualquer objeto quântico, deve-se recorrer à lógica e à matemática usuais, inclusive os métodos indutivos e sobretudo a estatística tradicional. Um dos objetivos deste artigo é o de mostrar que a afirmação contida neste último parágrafo é falsa.

As teorias da física clássica podem ou não incluir a validade da ação à distância. Por exemplo, a astronomia newtoniana, ainda em voga em muitos casos, aceita a ação à distância no caso da gravitação universal. Porém, na relatividade geral, isto não é admitido.

Dada teoria física diz-se local se nela não houver ação à distância; em caso contrário, ela é não local.

Em MQ tem-se a seguinte proposição:

Teorema 12.1. *Em MQ há ação à distância, ou seja, ela não é local. (Ver [9]).*

Outro fato relevante sobre a MQ é o seguinte: A MQ, fundamentada na equação de Schrödinger, é indecidível. Com efeito, vale a sentença:

Teorema 12.2 (Indicidibilidade da MQ)**.** *Não há máquina de Turing que decida se um sistema quântico tem 'gaps]' espectrais ou não [17].*

O teorema de Kochen-Specker, que foi descoberto independentemente por Bell, e que está estreitamente ligado ao deste último, é o seguinte:

Teorema 12.3 (Kochen-Specker)**.** *As diversas grandezas de um sistema quântico não se encontram simultaneamente definidas em todos os instantes do tempo ([81]).*

Corolário 12.4. *Em QM não há variáveis ocultas.*

Corolário 12.5. *O realismo quântico é parcial: as quantidades quânticas de um sistema quântico não possuem valores definidos antes da observação.*

Corolário 12.6. *A lógica da QM é uma lógica "parcial", que não se estende simultaneamente a todo o sistema físico, contrariamente ao que ocorre com as lógicas tradicionais. (Consultar [81] e [85].)*

Do teorema de Kochen-Specker decorre que as grandezas de um sistema quântico não se encontram definidas simultaneamente em todos os instantes do tempo e em todo o espaço de Hilbert associados ao sistema. Porém, a álgebra das grandezas definidas em dado instante do tempo e na vizinhança de um ponto do espaço de Hilbert em questão, compatíveis entre si, formam uma álgebra de Boole parcial ([81] e [85]). Logo isto evidencia que a estrutura algébrica dos observáveis de um sistema quântico é composta por uma família de álgebras de Boole parciais e não por uma única álgebra envolvendo, ao mesmo tempo, todas as grandezas quânticas. (Como se acreditava que fosse o caso.)

Os corolários 1, 2 e 3 mostram patentemente que as grandezas quânticas são muito diferentes das da física clássica. Em outras palavras, a física quântica diverge profundamente da física dos objetos macroscópicos do dia a dia. (Como P. Jordan afirmou: “As observações não somente alteram o que se vai medir, mas são suas causas.”)

O teorema de Bell é de 1964, embora sua primeira verificação seja de Aspect em 1982. Há outras confirmações posteriores, bem como generalizações da desigualdade de Bell, essecial para a demonstração do referido teorema, como a desigualdade CHSH (de Clauser, Horne, Shymony e de Holt). Tais verificações são sofisticadas e de difícil realização. Sobre o teorema de Bell e sua desigualdade, a dificuldade de se evitar *loopholes* e a verificação final, definitiva, dos resultados, ver [62].

1. Observações complementares

De todas as teorias da física, a MQ é, talvez, a que tem o maior número de aplicações revolucionárias, atuais e potenciais, como, por exemplo, nas seguintes áreas:

Teorias da informação
Teletransporte
Intercâmbio de entrelaçamento
Computadores quânticos
Quebra de códicos
Criptografia quântica (a arte de se criar textos cifrados que só quem enviou conhece e só quem recebeu pode decifrar)
Teorias de controle
Sistemas de dados
Tradução de textos
Medicina quântica
Engenharia de planejamento
Inteligência artificial
Análise conceitual
Engenharia de software
Processamento de sinais

Robótica
Controle de tráfico em grandes cidades
Computação neural
Teoria da decisão
Teoria de jogos
Finanças
Semântica formal
Linguística computacional
Engenharia paraconsistente (Ver S. Akama, editor, Towards Paraconsistent Engineering, Springer, 2016)
Geração de números aleatórios

Duas obras elementares muito interessantes sobre a mecânica quântica são as seguintes: 1) Polkinghorne, J. C., *The Quantum World*, Princeton, 1989. 2) Cassinelo, A. e Gómes, S., *O Mistério Quântico,* Editora Planeta, 2017. (Os artigos [62] e [17] são de divulgação, mas fazem referência a trabalhos inovadores, de pesquisa.)

13

ON PARTICLES

> A polaron is a quasiparticle of condensed matter physics, introduced by Lev Landau. A quasi-particle constitutes a virtual particle used to simplify the physics of certain macroscopically complicated systems; clearly, strictly speaking they don't exist.
> Lars Eriksen

The philosopher interested in the foundations of today physics has, sooner or later, to deal with the problem of the nature of particles. When the physicist talks about particles, is he talking about individuals, real metaphysical substances? Or is he making reference, basically, to other entities or theoretical constructions, for example fields?

My position is that particles aren't necessary to exist as ultimate constituents of the world. They are conceptual tools to take into account, in a simple way, of some aspects of certain physical systems.

The term "particle", usually, is employed in physics in an informal way, without a clear and determinate meaning. There exist important differences in the use of the term "particle" in classical physics, QM (non relativist quantum mechanics) and QFT (quantum field theory).

Loosely speaking, in classical physics, particularly in classical mechanics, a particle is always a small material body the degree of smallness being a function of the context. In some cases, a particle can be identified with a moving point, its real dimensions presenting no relevance (point, in this sense, can not be a geometrical point, since such points are necessarily fixed, etc.). Classical particles are stable, have mass and occupy a definite position in any given moment of time. Classical particles are connected with classical space-time. The letter separates the former in the sense that different particles can not possess the same space-time coordinates; if this happens, they are equal (or identical). Particles are individuals, subjected to the laws of classical mechanics (and to the norms of classical logic), specially those of Lagrangean and Hamiltonian mechanics.

In QM particles have, normally, small velocities as compared with the velocity of light. In other words they are not relativistic bodies (special relativity is not intended to apply to them). The central novelty in relation to classical mechanics is that in QM the dynamical quantities are governed by quantization and probabilistic principles. As a consequence, their behaviour, in some sense, is not deterministic, although Schrödinger equation is. In effect this equation is an equation of classical mechanics; only when it is combined with quantization and probability, we get the really the non- classical behavior and stance of QM.

The panorama changes completely in QFT (here, we take into consideration only the standard model). Particles becomes "modes" of "vibration" of fields, and radiation and interaction between particles are taken into account. When the degrees of freedom of fields are finite then it becomes possible to talk about such modes as if they were particles. The philosopher must be cautious in connection with this matter, because physicists commonly make reference to particles, say electrons and protons, when they should be more precise, talking about electron-fields and proton-fields. All the fields of QFT are renormalizable, renormalization being an operation or method of solving equations that is typical of this theory, and that is out of the domain of QM.

Dirac equation is a central equation of QFT. It rules over fermion fields. Dirac thought that his equation concerned particles, but finally arrived at the conclusion that it was a field equation. As Auyang writes:

> The Schrödinger equation is not relativistic. There are several relativistic quantum equations of motion; the most common ones are Klein-Gordon equation for spin zero systems and Dirac equation for spin 1/2 systems.

> In one interpretation, they are relativistic wave equations for single particles, so that their variables $\psi(x)$are single-particle wave functions in strict analogy with nonrelativistic quantum mechanics. This interpretation runs into two difficulties. First, unlike the Schrödinger equation, the Klein-Gordon equation does not give a positive-definite probability. Second, its solutions include states with negative energies. The negative-energy states make the system unstable in interaction, for the particles would continue to cascade toward the state with infinite negative energy, emitting infinite radioactive energy in the process. These difficulties force us to abandon the single-particle interpretation of the Klein-Gordon equation. The Dirac equation gives positive probability, but it too contains negative-energy states. To avoid the catastrophe of infinite decay, Dirac postulated that the negative-energy states are completely filled by other particles, which led to the prediction of a 'hole' in the field sea. The prediction of the 'hole' or the *antiparticle* as it came to be known, was a great triumph. However, it demands a many-particle picture in contradiction to the original single-particle interpretation. (Auyang [7], pp. 49-50)

Auyang adds that:

> The difficulties disappear if the variables $\psi(x)$ of the relativistic equations are interpreted not as single particle wavefunctions but as dynamic variables for continuous systems, or simply as fields. Field interpretation became the consensus among physicists. The discussion of two possible interpretations in textbooks do not imply the under-determination of theories; the single-particle interpretation does not work. (Auyang [7] p. 50)

So, if we believe that Dirac equation constitutes an equation governing particles, we have to try to eliminate the occurrence of negative energy states; however, it is not known how we should proceed to attain this objective.

The word particle in QM and in QFT has quite distinct meanings not always clear.

It is pertinent to note that in quantum statistics the usage of particles presents great practical advantages. For example, Bose-Einstein and Fermi-Dirac statistics consist in the introduction of appropriate probabilistic measures in certain "ensembles" as if they were really composed by particles. This is a procedure analogous to the use of QM in the dynamical description of atoms and molecules: a *façon de parler*, since at the bottom we have only fields and combination of fields.

In fact, one of the tasks of the philosopher of physics is to elucidate the various meanings of "particle" and the nature of the objects denoted by this term. Without such elucidation, it is difficult to treat most fundamental problems of today physics (for more details, see Falkenberg 2007).

My view concerning particles may be better understood with the help of the concept of soliton.

Some equations of propagation possess wave solutions which are stable, solitary and particle like. They appear in many situations in a variety of domains of physics, like fluid mechanics, plasma and lasers. Arnold summarizes the subject as follows:

> Not all first integrals of equations in classical mechanics are explained by obvious symmetries of a problem (examples are specific integrals of Kepler's problem, the problem of geodesics on an ellipsoid, etc). In such cases, we speak of "hidden" symmetry.

Interesting examples of such hidden symmetry are furnished by the Kortweg-de Vries equation:

$$\partial_t \phi + 6\phi\, \partial_x \phi + \partial_x^3 \phi = 0,$$

This nonlinear partial differential equation first arouse in the theory of waves in shallow water; later it turned out that this equation is encountered in a whole series of problems in mathematical physics.

As a result of a series of numerical experiments, remarkable properties of solutions of this equation with zero boundary conditions at infinity were discovered: as t tends to infinity or t tends to minus infinity these solutions decompose into "solitons" — waves of definite form moving with different velocities.

When solitons collide, there is a complicated interaction. However, numerical experiments showed that the sizes and velocities of the solitons do not change as a result of collision. (Arnold [5], p. 453).

Solitons behave like particles, but are in effect waves. Two states of the "same" soliton (wave) at different times, therefore, can not be said to be equal. The particle aspect does not entail a strict substantial or individual nature.

The QFT particles also behave like particles having substantial dignity, at least under certain circumstances. Nevertheless, they constitute "modes" of fields.

Final remarks

1. The concept of particle is not well defined; among other things, it depends on the theory one is considering.

2. There are two basic types of definition: the implicit, by the means of a system of postulates which does permit the characterization of a term or a concept, and the nominal definition when a new symbol is introduced via an explicit definition. Usually, in physics, the notion of particle is at best treated implicitly, with the help of a system of axioms that is at best only outlined.

3. Each time a theory is expanded or modified, the meanings of its implicit definitions are ipso facto altered. However, this state of affairs is not in general taken into account as it should be, especially in everyday physics.

4. The logical theory of definition and its applications are usually not discussed in physics, what causes a great number of difficulties.

5. In particular, the common theories of physics should be better analyzed from the logical point of view if one wants to arrive at a reasonable understanding of the notion of particle.

Remark [4], [61] and [83] constitute excellent works on quantum field theory; [7] and [51] are basic for our exposition; [5] is one of the best expositions of classical mechanics, containing a short but interesting discussion of solitons.

14

SET THEORY AND QUANTUM MECHANICS (SUMMARY)

THE OBJECTIVE of this paper is to outline, in an informal way, a rigorous set-theoretic basis for Quantum Mechanics (QM). Details are left for the reader.

I

Here, QM will be a non relativistic quantum mechanics. It is founded on classical set theory and Galileo-Newton space-time, having several empirically equivalent forms (see [96]). In one of these formulations, here adopted, its mathematical counterpart is essentially constituted by the theory of separable Hilbert spaces plus the Schrödinger's equation for finite sets of particles.

Our set theory is Zermelo-Fraenkel theory with a set U of Urelemente, denoted by ZFU. U is composed by U_0, the set of common Urelemente, plus the union of a finite class of disjoint sets, $\{U_1, U_2, \ldots, U_n\}$, n being a finite ordinal. In other words, U is the union of $U_0, U_1, U_2, \ldots, U_n$, any two of which are disjoint, and we put $U' = U - \{U_0\}$; this set is the collection of quantum Urelemente.

We extend ZFU by the introduction of Galileo-Newton space-time in such a way that all Urelemente are immersed into the space-time variety.

If x, y belong to U', x and y are said to be *equivalent* and we write $x \mathtt{eq}\, y$, if, for some natural number m, $0 < m < n+1$, x, y belong to U_m.

It is easy to show that if x, y and z belong to U' , then: $x \text{ eq } x$; $x \text{ eq } y \rightarrow y \text{ eq } x$; $(x \text{ eq } y \wedge y \text{ eq } z) \rightarrow x \text{ eq } z$, and $x = y \rightarrow x \text{ eq } y$.

We need an additional postulate called Principle of Quantum Substitutivity:

Let z be a variable for Urelemente, $F(z)$ a formula in which z occurs free and is not subjected to any non-trivial space-time restrictions. Under these conditions, if x and y are variables for quantum Urelemente, both free for z in $F(z)$, then:

$$x \text{ eq } y \rightarrow (F(x) \leftrightarrow F(y)).$$

The symbol of identity may be interpreted as the usual identity or as a relation of equivalence satisfying an axiom of substitutivity. On the other hand, $x \text{ eq } y$, defined between Urelemente x and y, means that x and y have the same quantum properties, leaving aside the space-time properties.

There are two kinds of logic in the present-day physics: the standard one, of the macroworld, and the quantum logic of the microworld. The first is the usual mathematical logic, here already delineated. The second is a kind of local logic, a form of the so called quantum logic, which will become clearer in what follows.

II

To exemplify the ideas obove, let us consider the treatment of the logical foundations of QM exposed by Manin in his book [85], pages 78-89. The author writes on page 83:

> We now consider the language of quantum mechanics, oriented on describing a system S. We shall exclude the time aspect by fixing a moment of time to which all statements about the state of the system refer. Then the 'state of the system' will be the only variable in the language. It takes values in the set of lines in the Hilbert space HS. The only questions to which we can give a yes or no answer are those of the form: 'Does the state of the system belong to a given closed subspaece of HS?' It is the closed subspaces of HS that form the analogy of the Boolean algebra B. The conjunction of questions corresponds to the intersection of subspaces, and the disjunction corresponds to their sum, but both operations can be performed only when the corresponding

> projection observables commute. Only in this case are the Boolean identities fulfilled.

Then Manin presents the concept of a partial Boolean algebra in order to make an algebrization of what we may call the partial algebra of quantum propositions (cf. [85], pages cited at the beginning of this section). This way, we get an algebrization of the logic of quantum propositions that give us a really good idea of the inner structure of QM (more details will be presented in a future work; in particular, we shall formulate a propositional version of the particular partial álgebra of the quantum propositions).

It should be clear that all constructions of Manin's exposition can be reproduced in ZFU. (ZFC consists in a modification of the set theory presented in [14].)

III

The concept of identity is one of the motifs of various disagreements in QM. For example, taking into account the Bose-Einstein and the Fermi-Dirac condensates, it seemed that identity cannot be applied to elementary particles.

However, Dehmelt and collaborators showed, in 1973, that electrons and positrons, for example, could be isolated and studied individually in their behaviour, establishing that elementary particles are subjected to the category of identity (even a weak kind of identity, whose applicability is related to space and time).

We will not consider here the theory of condensates of quantum particles, which is well-known (since it is the result of the fact that, leaving aside space-time, two quantum particles of the same species are indistinguishable).

The central event here is the following: it was discovered that it is possible to isolate a quantum particle to study it and make experiments with it, as it was shown by Dehmelt and associates, contrary to the general view of physicists of his epoch. In particular that quantum particles possess a class of identity, even of a weak identity, that can be made explicit only through space and time. Making a parody of Kant, we could assert that space and time were given to us specially to help our capacity of distinguishing quantum particles.

Two relevant questions of the history of QM are the following: the problems of the relation of equality of quantum particles and the one of the meaning of individuality of such particles. We shall not discuss these topics here, but only remark that they can be treated employing the set-theoretic foundations here proposed for such mechanics: they simply don't appear in the more rigorous formulations of the subject (see, for example, [59] and [70]). We merely proceed to outline how these themes are dealt with within ZFC.

Concerning the themes of identity and of individuality, the sole that interest us by now, it is enough to reproduce the following passage of Wick [82], p. 138:

> Dehmelt's interest in confining an elementary particle had been piqued in his student days in Germany, when his teacher drew a dot on the blackboard and declared 'Here is an electron. . . ' It had been thought since Dirac invented his relativistic wave equation in the 1930s that the electron had no structure at all and was literally a mathematical point. Capturing and controlling this most infinitesimal entity would represent a coup de maitre of experimental physics. With D. Wineland and P. Ekstrom, Dehmelt first bagged a solitary electron in 1973, trapping it in an invisible cage of electric and magnetic fields. But a few more years were to pass before he and collaborators in Seattle learned to cool down and interrogate their prisionner. Once that was accomplished, they could study it at their leisure.

Wick also writes that ([82], p. 139):

> [in 1984], Dehmelt, working with R. S. Van Dyck, Jr, and P. Schwinberg, had performed the even more astonishing trick of confining a positron, the antimatter twin of an electron, for a period of months. Their positron originates in the decay of a radioactive nucleous, and they could be sure of its identity during those many days since it had no opportunity to exchange places with another positron. As time went by and the caged particle continued in existence, its spin-flips registering on their devices like so many turns of the wheel in a hamster's cage, they began to think of it as a pet – and, as one does for pets, gave it an affectionate moniker ("Priscilla"). The ghosts of Schrödinger and Mach had been properly propitiated.

In 1989, Dehmelt received the Nobel Prize for his work. From his labor, we arrive at the solutions of the two problems referred to at the beginning of this section...

IV

The works of Serge Haroche and of David J. Wineland, on particle control in the quantum world, confirm our conclusions of this paper. Haroche and Wineland received the Nobel Prize in 2012 for their works on particle control (see [101]).

V

Final note:

By the means of the set-theoretic base above delineated and the use of our treatment of space-time in QM, also described above, it is apparently possible to overcome other difficulties concerning QM such as the doubts connected with the double-slit experiment and with the quantum concept of superposition. Sometimes, we have to extend the described QM by the addition of new postulates and notions, for example in relation with the double-slit experiment. We shall justify and explain this procedure in future articles.

15

SOBRE A CONJETURA DA MASSA POSITIVA

> Mathematics is often called the language of science, or at least the language of the physical sciences, and that is certainly true: Our physical laws can only be stated precisely in terms of mathematical equations rather than through the written or spoken word. Yet regarding mathematics as merely a language doesn't do justice to the subject at all, as the word leaves the erroneous impression that, save for some minor tweaks here and there, the whole business has been pretty well sorted out.
> Shing-Tung Yau

Este capítulo é formado de várias observações sobre o assunto do título.

1) A conjetura da massa ou da energia positiva é uma das questões mais debatidas em física. Como vamos nos limitar à relatividade geral (RG), tanto faz falar de massa como de energia. A conjetura é: *A massa (ou a energia) de um sistema físico isolado é positiva (estritamente maior do que zero). Em particular, isto se aplica ao universo como um todo.*

2) O espaço-tempo da RG não pode ser estável se a conjetura não se verificar para o universo global.

3) Sabe-se que, se a densidade média de matéria for positiva, então a curvatura média é positiva e reciprocamente.

4) A equação de Einstein da RG conecta física e geometria. Por isso, certas questões de física da RG interessam tanto aos geômetras.

5) A densidade de matéria de nosso universo é positiva (estritamente maior do que zero), mas a massa total do mesmo poderia se negativa (pela gravitação).

6) A conjetura da massa ou energia positiva foi provada Shing-Tung Yau e Richard Schoen em 1979. Edward Witten deu outra demonstração completamente diferente em l981.

7) Dos trabalhos desses matemáticos e físicos, bem como de outros, resulta que o universo em RG é estável. (Se a energia é positiva, então há um limite inferior para ela: zero, a energia do vácuo. Em caso contrario, não haveria tal limite e o vácuo poderia ter energia negativa cada vez menor.) Mas a noção de massa em RG é "ilusória", difícil de se definir localmente (embora globalmente não haja grandes problemas).

8) No tocante a buracos negros, vale o teorema de Hawking-Penrose, que é o seguinte: Se existem certas superfícies, as chamadas superfícies *trapped*, então existem singularidades (e buracos negros). Ademais, também se tem o teorema de Yau-Schoen:

Theorem 15.1 (Yau-Schoen)**.** *Quando a densidade de uma região chega a certo limite (uma ou duas vezes a densidade de uma estrela de neutrons), então existe uma superfície trapped e se forma necessariamente um buraco negro.*

9) Na mecânica clássica a conjetura da massa positiva, acima referida, é obviamente satisfeita. E não há buracos negros.

10) É preciso que se verifique, exatamente, quais são as suposições do teorema da energia positiva. Witten [105] escreveu:

> Had the positive mass theorem been untrue, this would have drastic implications for theoretical physics, since it would mean that conventional space-time is instable in general relativity.

11) Qualquer trabalho sobre a energia ou a massa do universo, em nossa opinião, deveria fazer menção à conjetura da energia positiva.

12) Observação. A seguinte citação do livro de Yau & Nadis, relacionado na bibliografia, é relevante:

> Equipped with Riemann's metric tensor, Einstein worked out the shape and other properties — the geometry, in other words — of his newly conceived space-time. And the resulting synthesis of geometry and physics, culminating in the famous Einstein field equation, illustrates that gravity — the force that shapes the cosmos on the largest scales — can be regarded as a kind of illusion caused by the curvature of space and time. The metric tensor of Riemannian geometry not only described the curvature of spacetime but, also described the gravitational field in Einstein's new theory. Thus, a massive body like the sun warps the fabric of spacetime in the same way that a large man deforms a trampoline. And, just as a small marble thrown onto the trampoline will spiral around the heavier man, ultimately falling into the dip he creates, the geometry of warped spacetime causes Earth to orbit the sun. Gravity, in other words, is geometry. The physicist John Wheeler once explained Einstein's picture of gravity this way: "Mass grips space by telling it how to curve; space grips mass by telling it how to move."

Nota Sobre as duas citações em inglês, ver Shing-Tung & Nadis, , páginas ix e 32.

Merece menção o fato de que Yau também demonstrou a chamada conjetura de Calabi, provando que as equações de Einstein do vácuo tem solução não trivial (Shang-Tung & Nadis, capítulo 5).

16

NOTAS SOBRE A MECÂNICA QUÂNTICA E SOBRE AS TEORIAS QUÂNTICAS DE CAMPOS

NESTE CAPÍTULO, "MQ" designará a mecânica quântica não relativista, em qualquer de suas formulações, e "QFT" designará uma teoria quântica de campos ou *as* teorias quânticas de campos em geral. O contexto deixará claro ao que estaremos nos referindo especificamente.

David Griffiths, em seu excelente livro de mecânica quântica, escreveu o seguinte:

> Unlike Newton's mechanics, or Maxwell's electrodynamics, or Einstein's relativity, quantum theory was not created —or even definitively packaged— by one individual, and it retains to this day some scars of its exhilarating but traumatic youth. There is no general consensus as to what its fundamental principles are, how it should be taught, or what it really 'means'. Every competent physicist can 'do' quantum mechanics, but the stories we tell ourselves about what we are doing are as various as the tales of Scheherazade, and almost as implausible Niels Bohr said, 'If you are not confused by quantum physics then you haven't really understood it'; Richard Feynman remarked: 'I think I can safely say that nobody understands quantum mechanics'. ([59] página vii).

Neste trabalho, apresentamos comentários sobre a MQ (mecânica quântica não relativista) e sobre a QFT (teoria quântica de campos, na sigla em inglês).

Duas definições preliminares:

1) Uma teoria com variáveis ocultas é uma teoria com modelos probabilísticos tal que, se a ela adicionarmos novas variáveis correspondentes a certas grandezas que nela não figuram, a transformamos em teoria determinística.

2) Na física (em particular na MQ), um teorema 'no-go' afirma que determinada situação não é possível do ponto de vista físico (do prisma da MQ).

Consideraremos, agora, alguns dos mais importantes teoremas 'no-go' da MQ (ver [9], [16], [59] e [81]).

Teorema de Bell. *Nenhuma teoria de variáveis ocultas locais (não há ação à distância) pode reproduzir as predições da MQ.*

Outra versão do teorema de Bell — A MQ não pode ser considerada uma teoria de varáveis ocultas locais, mantendo-se suas previsões empíricas; isto significa que ela implica ação à distância.

Dada a relevância do teorema de Bell, reproduziremos, a seguir, duas formulações do próprio Bell ([9]).

Primeira: "In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that the theory could not be Lorentz invariant."

Segunda: "If the quantum mechanical formalism is correct, then the system consisting of a pair of entangled electrons cannot satisfy the principle of local realism."

Outro 'no-go' teorema muito significativo é o de Kochen-Specker:

Teorema de Kochen-Specker. *Dado qualquer instante do tempo, em um sistema quântico há sempre grandezas que não possuem valor definido.*

A proposição anterior acarreta que os observáveis quânticos não possuem valores definidos em todos os instantes do tempo. Por outro lado, quando existem, esses valores são intrinsicamente definidos, isto é, independem do processo utilizado para defini-los.

Isham e Butterfield afirmam que o teorema de Kochen-Specker "... asserts the impossibility of assigning values to all physical quantities, whilst, at the same time, preserving the functional relations between them."

De fato, os teoremas de Bell e de Kochen-Specker acham-se intrinsicamente relacionados, e o segundo constitui uma espécie de complemento do primeiro.

Outro resultado que merece destaque é o chamado *The free will thorem* de Conway-Kochen. Nas próprias palavras dos autores, ele pode ser assim enunciado:

Teorema de Conway-Kochen. *The response of a spin 1 particle to a triple experiment is free —that is to say, is not a function of properties of that part of the universe that is earlier than this response with respect to any given inertial frame (ver [16]).*

O teorema precedente é muito discutido e pode ser defendido como um 'no-go' teorema no tocante ao determinismo (David Hodgson).

Observações:

1) Griffiths, em [59], expõe temas de grande relevância para o filósofo da física, tais como, por exemplo, os seguintes: o paradoxo EPR, o teorema de Bell, o 'no-clone theorem', a questão do gato de Schrödinger e o paradoxo de Zenão quântico. Todos esses assuntos são de enorme interesse para todos que desejam "penetrar nas brumas da MQ".

2) Não é possível se observar a onda da equação de Schrödinger; será ela uma entidade metafísica? Recorde-se que ela está no espaço de Hilbert e não no espaço-tempo 'real'.

3) Um estado de superposição também é, em princípio, não observável? Quando há superposição, onde se encontra a partícula correspondente?

4) O paradoxo do gato de Schrödinger deve ser interpretado como provando a existência de superposição de estados de objetos macroscópicos ou como a prova de que superposições de tais objetos não existem?

5) O colapso da função de onda não é consequência da equação de Schrödinger, constituindo-se em algo *had hoc*; em que ele consiste efetivamente?

6) Uma nova maneira, mais precisa, de se enunciar o teorema de Kochen-Specker é a seguinte: —Seja S um sistema quântico cujo espaço de Hilbert tem dimensão maior ou igual a 3. Então, as grandezas de S não se encontram simultaneamente definidas em todos os instantes do tempo." Por quê?

Métodos de aproximação acham-se intimamente ligados à MQ sobretudo no tocante à solução aproximada da equação de Schrödinger, métodos que, em boa parte, se originaram no âmbito da mecânica celeste. Em particular, o método das perturbações desempenha um papel significativo na MQ. A essência deste método é a decomposição de um sistema quântico que se investiga em duas partes, uma para a qual se pode obter soluções sem grandes dificuldades, e um termo 'extra', que complica o cálculo das soluções. Então, a teoria das perturbações permite que se obtenha o termo extra por meio de séries de potências. Cada termo da correção se calcula por esse método. O primeiro termo calculado é uma correção da solução aproximada inicial e os seguintes corrigem a soma dos já calculados.

Sem os métodos de aproximação, em particular do método das perturbações, não há, em certo sentido, MQ. As soluções efetivas da equação de Schrödinger dependem essencialmente desses métodos. Poder-se-ia asseverar que tais métodos pertencem à própria natureza da MQ. (Quase a metade do livro de Griffiths [59] é devotado aos métodos aproximados da MQ.)

Note-se, de passagem, que a própria utilização de diagramas de Feynman em MQ constitui uma visão 'geométrica' dos processos de aproximação da mesma. Por outro lado, o emprego de séries de potências e de renormalização em QFT acha-se conectado, ainda que indiretamente, aos métodos de aproximação da MQ.

A exposição anterior torna patente que a MQ difere profundamente da maneira clássica de se fazer física, inclusive da postura científica da relatividade geral.

Pode-se, na realidade, sustentar que a física tradicional busca a verdade, que jaz sob seu desenvolvimento incessante. Trata-se aqui, evidentemente, da verdade como correspondência. Porém, acreditamos que, ao contrário, a MQ quebra tal esquema, implícita ou explicitamente: ao que tudo parece indicar, ela foi forçada a se apartar dessa finalidade. Aos poucos, percebe-se que ela aspira, basicamente, a quasi-verdade, como a definimos em [37], isto é, principalmente, salva as aparências. Ou seja, ela busca a verdade pragmática como se define em [37]. Daí, sua finalidade de se procurar construções teóricas que sistematizam a experiência com o objetivo de prever resultados experimentais e observacionais ('prever para poder').

O caráter de perseguir a quase-verdade explica certos traços que lhe são peculiares: formulações diversas embora empiricamente equivalentes; utilização do conceito de probabilidade como ferramenta central de previsão; métodos diversos para a solução da equação de Schrödinger; soluções unicamente aproximadas em geral; adaptabilidade às mais variadas circunstâncias empíricas (aplicações em eletrônica, semicondutores, chips de silício, computadores, telefones celulares, jogos eletrônicos, circuitos integrados, transistores, lazer, equipamentos CD, etc.).

Ademais, serve como ponto de referência de outras teorias científicas, tais como a teoria quântica de campos, a mecânica quântica relativista especial, etc.

A seguir, discorremos sobre diversos tópicos de MQ, em forma de proposições:

1) A MQ pode ser apresentada via os mais variados formalismos matemáticos que, do prisma das aplicações, conduzem, praticamente, aos mesmos resultados empíricos, embora não sejam matematicamente equivalentes. Por exemplo, os seguintes: a formulação matricial de Heisenberg, a ondulatória de Schrödinger, a da integral de Feynman e a via 'rigged Hilbert spaces' (ver Styer e outros [96]).

2) No tocante à observação e à experimentação, nucleares para a física, sobretudo para se efetivar previsões e retrovisões, o espaço e o tempo são imprescindíveis; em particular, sem espaço e sem tempo não se pode testar uma teoria da física. Mais ainda, a idealização de situações empíricas, por meio de diversos tipos de espaço-tempos, é fundamental.

3) Os espaço-tempos mais utilizados em física são o clássico de Galileo-Newton, o de Minkowski da relatividade restrita, e o da relatividade geral (espaço de Riemann). Há outros menos utilizados, tais como as variedades de Callabi-Yau, relacionadas com a teoria das cordas.

4) A MQ fundamenta-se no espaço-tempo clássico, que é o produto do espaço euclidiano tridimensional pela variedade temporal, composta pelos

números reais. A questão central, então, converte-se no estudo das propriedades mais relevantes do espaço-tempo clássico (topologia, teorema de Cantor sobre o cardinal do espaço e do tempo, etc.) e da lógica subjacente à teoria. (Será, esta lógica, a clássica ou alguma outra não clássica?)

5) Quando se faz referência à lógica da MQ é preciso que fique claro o que tal expressão significa. Há duas interpretações canônicas: ela se refere ao reticulado dos observáveis da MQ ou à lógica dedutiva que serve de alicerce ao desenvolvimento dessa teoria. Em geral, a que nos interessa, no momento, é a lógica neste segundo sentido.

6) No caso bem conhecido da mecânica clássica, o reticulado dos observáveis é uma algebra de Boole. Na MQ, tem-se uma estrutura não booleana (reticulado ortomodular, algebras de Boole parciais, . . .).

7) Um ponto fundamental com referência à MQ, que merece mais cuidado e exames minuciosos é o de se compatibilizar as duas afirmações seguintes: a) A álgebra dos observáveis não é booleana; b) A lógica subjacente à MQ é de natureza clássica: cálculo de predicados de primeira ordem mais uma teoria de conjuntos conveniente (ou algum sistema equivalente).

8) Alguns aspectos da MQ que normalmente não são debatidos e tornados explícitos são os seguintes:

1. A MQ supostamente se funda na lógica clássica, embora haja algumas formulações em que isso não é verdadeiro, tais como certas sistematizações paraconsistentes e outras polivalentes. Mas, quando se muda de lógica, a MQ permanence a mesma?

2. Há restrições quanto à MQ: ela se baseia no espaço-tempo tradicional de Galileo-Newton, ao contrário de outras teorias da física, como a relatividade geral, cujo espaço-tempo subjacente é riemanniano, não euclidiano; como se explica isso? Ademais, nela não há choque de partículas nem se pode levar em conta a teoria da radiação; estes fatos não restringem demais o domínio da MQ?

3. Existem diversas versões da MQ, em muitos casos envolvendo idéias metafísicas. Todas elas, ao que tudo parece indicar, conduzem aos mesmos resultados empíricos. Como se pode (ou se deve) escolher uma dentre elas como sendo a verdadeira?

> Do prisma científico, elas são todas quase-verdadeiras e, da perspectiva metafísica, há várias posições possíveis. (Quiçá a física e a ciência empírica em geral busquem tão somente a quase-verdade (ver [37]).

9) A identidade, em MQ, tem sido muito debatida. Ela claramente apresenta problemas em MQ, sobretudo com referência a seu significado (pode ter uma conotação metafísica, ser tratada apenas como um relação de congruência apropriada, vista tão somente como abreviação eliminável, . . .). Um exemplo encontra-se no excelente livro de Griffiths [59], no qual o autor encontra dificuldades para tratar das estatísticas de Bose-Einstein e de Fermi-Dirac (consultar, também, French e Krause [56]).

10) É possível edificar as formulações mais comuns da MQ nas teorias de conjuntos usuais, com certas reservas motivadas pelo teorema de Kochen-Specker.

11) É possível, também, fundamentar a MQ em lógicas não-reflexivas; é o caso de da Costa e de Ronde [44], para citar um exemplo. No entanto parece que se tem de reelaborar o espaço-tempo, distinguindo-se propriedades intrínsecas das extrínsecas; isto seria feito de maneira a se poder 'separar' determinados objetos quânticos.

12) Tem-se: a) As grandes teorias da física, supostas quase-verdadeiras, não são nem verificáveis nem refutáveis de modo definitivo e completo. 2) Não há propriamente refutação possível da quase-verdade de uma teoria, porém, sim, restrições em seu domínio de aplicação. 3) As definições metodológicas tradicionais não têm aplicação propriamente dita na física hodierna, como, digamos, a noção padrão de teoria (em lógica). 4) Em MQ, o próprio conceito tradicional de experiência parece não funcionar *in totum* (por exemplo, não se refuta definitivamente a quase-verdade de uma teoria, mas pode-se apenas restringir seu campo de aplicação).

Na teoria quântica de campos (QFT) a noção fundamental é de de campo. Os campos, em QFT, são funções do espaço-tempo com valores escalares, vetoriais, tensoriais ou consistem em operadores, sendo estes últimos característicos da QFT. Por outro lado, convém deixar claro que há diversas QFTs que não são equivalentes entre si; todavia, este fato ficará em geral subentendido na presente exposição.

As partículas serão consideradas, como é usual, estados excitados de campos. Normalmente, o espaço-tempo da QFT é o da relatividade restrita, o espaço-tempo de Minkowski (em todo caso, espaço-tempos não relativísticos podem ser empregados). A QFT não se aplica apenas a partículas assim concebidas e que dão conta dos fenômenos microscópicos e atômicos, mas também às quase-partículas que ocorrem em matéria condensada.

Uma das vias para a edificação da QFT é a teoria lagrangeana de campos, complementada com a quantificação de campos (operação também chamada de segunda quantização). Exemplos são as teorias do chamado modelo padrão: QED (força electro-magnética), QCD (força forte) e QFD (teoria electro-fraca, que trata simultaneamente das forces fraca e electro-magnética); a força fraca também tem uma teoria própria que podemos chamar de teoria V-D (elaborada por Sudarshan e outros). Há, ainda, o campo gravitacional.

Na teoria de campos não existe ação à distância. Logo, a gravitação, ao contrário do que realizou Newton, não é uma força local, existindo ação à distância. Por conseguinte, a teoria de Newton é substituída, na física atual, por outra que não envolve tal ação: a relatividade geral.

Assim, na física atual, as causas de forças são as seguintes, que envolvem todas as forças conhecidas da natureza: o campos gravitacional, electromagnético, da força fraca e da força forte. Normalmente, os campos do segundo e do terceiro tipos são tratados juntamente.

É óbvio que um campo de operadores necessita que exista um outro campo básico onde atuarão os operadores.

Auyang escreve o seguinte:

> According to the current standard model of elementary particles physics based quantum field theory, the fundamental ontology of the world is a set of interacting fields. Two types of fields are distinguished: matter fields and interaction fields. Their general properties, including spin statistics, differ widely. (Auyang [7])

Uma teoria de campos como as que foram citadas acima envolvem a idéia de grupos de gauge, isto é, de grupos de simetria locais para campos e potenciais. No caso da QED, o grupo é comutativo e nos demais citados precedentemente, não é; neste caso, a teoria do gauge denomina-se teoria de Yang-Mills. Nestas teorias do modelo padrão, a interação entre partículas é explicada por meio de partículas mediadoras, como se sabe. Todas as interações (mecânicas ou não) reduzem-se a interações entre campos como, por exemplo, a superposição de campos.

Não obstante, não entraremos em mais detalhes sobre a teoria de campos que, de fato, em física, envolve quantização de campos, pois a MQ (que agora é a teoria que nos interessa) não é desse tipo de teoria. No entanto, ela é essencial para se conectar as teorias de campos a nosso contorno; a gravitação, por seu turno, ainda não é uma teoria quântica de campos, embora já se esteja pensando em gravitação quântica . . .

Em certo sentido, há incompatibilidade entre as teorias de campos usuais e a MQ. Em particular, o espaço-tempo das primeiras é o de Minkowski, ao passo que o da MQ é o tradicional de Galileo-Newton. Ao pé da letra, a MQ é logicamente incompatível com as teorias de campos acima referidas. Chega-se, então, a um problema sério: como se pode compatibilizar essas duas categorias de teorias que são, efetivamente, inconsistentes?

Parece que existem duas possibilidades de se solucionar semelhante problema;

1) Recorrer-se a uma lógica paraconsistente;

2) Considerar-se a MQ como uma espécie de aproximação de uma teoria de campos, pelo menos dentro de certas circunstâncias apropriadas.

O surpreendente é que ambas as soluções são logicamente possíveis, sendo a segunda a seguida. Pode-se tentar a primeira porém a segunda é a aceita, pois tem-se evidenciado ser suficientemente satisfatória. Por isso, a segunda possibilidade afigura-se remota, embora não impossível. Porém, a possibilidade de uma solução paraconsistente, do prisma filosófico, é de fundamental significação.

17

OBSERVAÇÕES SOBRE A EQUAÇÃO DE SCHRÖDINGER E OS FUNDAMENTOS DA MECÂNICA QUÂNTICA (RESUMO)

I think I can say that nobody understands quantum mechanics.
Richard Feynman

A equação de Schrödinger é a equação fundamental da QM (mecânica quântica não frelativista). Ela pode ser escrita assim:

$$i\hbar\frac{\partial\psi}{\partial t} = \mathsf{H}\psi, \tag{17.1}$$

onde H e ψ são, respectivamente, o hamiltoniano e a função de onda de um sistema quântico S, supondo-se que o sistema está isolado de quaisquer influências vindas de fora de S. Diz-se, então, que S é um sistema isolado (i e $\hbar$ possuem os significados usuais: i é a raiz quadrada de -1 e $\hbar$ é a constante de Plank dividida por 2π).

Um dos princípios básicos da física, em especial da QM, é o de que existem sistemas que podem ser tratados independentemente de quaisquer outros, isto é, que se encontram isolados. Estimar que certos sistemas encontram-se isolados é um dos procedimentos essenciais da física: isolar para vencer, como diria Cesar. Sem tal princípio não existiria a ciência empírica, de caráter experimental. Note-se, porém, que se trata de um isolamento metodológico sem nenhuma conotação ontológica.

Para uma partícula quântica ou para um sistema quântico de partículas, a equação de Schrödinger assume várias formas possíveis ([70] e [75]). Facilmente se demonstra que a função de onda e o hamiltoniano determinam, em certo sentido, o potencial quântico.

Um dos caminhos para se tratar da QM é o de se investigar, em detalhe, a equação de Schrödinger (ver [59] e [70]). Outra via consiste em se recorrer à teoria dos espaços de Hilbert, como von Neumann evidenciou ([75] e [102]). Neste segundo caso, associam-se aos sistemas quânticos espaços de Hilbert apropriados e, a partir desse processo, trata-se de se estabelecer uma teoria desses sistemas, a qual engloba os resultados obtidos de maneira mais direta.

Em outras palavras, para se investigar os sistemas quânticos de acordo com a segunda via descrita, associam-se espaços de Hilbert a esses sistemas, que, por assim dizer, dão conta de seus estados e permitem seu estudo. De uma certa perspectiva, os espaços de Hilbert retratam, matematicamente, os sistema em consideração.

Dois sistemas quânticos S1 e S2 com seus espaços de Hilbert correspondentes H_1 e H_2, são quanticamente independentes quando não interagem entre si, cada um tendo sua equação de Schrödinger e seu espaço de Hilbert próprios, sem nenhuma conexão quântica entre eles. Por outro lado, quando S1 e S2 interagem, formam, por assim dizer, um único sistema composto que tem como espaço de Hilbert associado o produto tensorial de H_1 por H_2. É este o novo sistema que se investiga em MQ em vez de H_1 e de H_2 separados (ver [75] e [102]).

Formularemos, a seguir, algumas questões sobre a QM:

1) A QM tem como equação básica a equação de Schrödinger. Nela aparece a unidade imaginária i. Como isto é possível? Esta unidade corresponde a alguma propriedade física real? Qual o significado deste artifício?

2) A referida equação é tal que não se sabe como integrá-la de modo explícito a não ser em casos especiais: poço infinito (*infinite square well*), oscilador harmônico, átomo de hidrogênio e partícula livre. Em geral só pode ser resolvida aproximadamente. Isto é lícito?

3) Não se conhecem métodos para se provar um teorema geral de existência e de unicidade para as soluções da equação de Schrödinger. Assim, sera justificável utilizá-la?

4) Diz-se que a equação em apreço é uma equação de ondas. Todavia, isto é estritamente falso. A equação de ondas de D'Alembert é (em mecânica dos fluidos), com referência da ondas unidimensionais a seguinte:

$$\frac{\partial^2 u}{dt^2} = a^2 \nabla^2 u. \tag{17.2}$$

Logo, como a equação de Schrödinger pode ser uma equação de ondas?

5) Para se resolver a referida equação básica da MQ há vários métodos tais como, por exemplo, os seguintes: por meio da teoria da perturbação, pelo princípio variacional, via a aproximação WKB (Wantzel, Kramer e Brillouin) e por intermédio da aproximação adiabática. Como se justificam tais métodos?

6) A MQ é, talvez a mais importante disciplina da física pelas suas aplicações, as mais variadas possíveis.

1. Postulados informais para a MQ

Postulado 1: O estado de um sistema quântico S é dado por um vetor de um espaço de Hilbert complexo Hs, que é o espaço de estados de S. As predições de resultados de medidas em S são fornecidos, probabilisticamente, pelos vetores de Hs, que, em certo sentido, representam os estados do sistema.

Postulado 2: Os observáveis de S são representados matematicamente pelos operadores auto-adjuntos de S.

Postulado 3: Sejam A uma quantidade observável, F um estado (vetor normalizado) e Â o operador auto-adjunto correspondente a A; então o valor esperado medindo-se A é:

$$\langle A \rangle_F = \langle F | \hat{A} | F \rangle.$$

Postulado 4: Em um sistema isolado, vale a equação de Scrödinger escrita no início deste artigo.

Postulado 5: O espaço de Hilbert de um sistema S composto por dois outros sistemas, S1 e S2, é o produto tensorial do espaço de Hilbert S1 pelo espaço de Hilbert S2 (e o processo se generaliza para qualquer número natural *n*, com *n* maior do que 2.

Sobre os postulados acima, consultar [70], que é a obra da qual tiramos os mesmos, com algumas adaptações óbvias. No referente à história da MQ vale a pena ler [104].

Finalmente, uma questão adicional. Em seu livro *Equações que Mudaram o Mundo* (Zahar, 2013), Ian Stewart escreve:

> "... a mecânica quântica seria apenas uma versão enfeitada da equação clássica de onda, resultando em duas ondas em vez de uma, se não houvesse um detalhe intrigante. Pode-se observar ondas clássicas, e ver qual é sua forma, mesmo que sejam superposições de diversos modos de Fourier. Mas na mecânica quântica, jamais se pode observar a função de onda inteira. Tudo que se pode observar numa dada ocasião é uma única autofunção componente. Grosso modo, se você tentar duas dessas componentes ao mesmo tempo, o processo de medicão em uma delas perturba a outra." E Stewart acrescenta: "Isso levanta imediatamente uma questão filosófica difícil. Se não se pode observar a função de onda inteira, será que ela realmente existe?" Como se explica isso?

2. Sobre as aplicações da QM

As aplicações da QM (um tanto ampliada para poder se envolver com determinadas areas da física com ela conectadas, como a eletrônica) são fantásticas, às vezes quase inacreditáveis. A seguir chamaremos a atenção para algumas dentre elas:

Eletrônica
Semicondutores
Circuitos integrados (chips de silício ...)
Computadores
Telefones celulares
Equipamentos CD e DVD
Jogos eletrônicos
Automóveis e aviões

Refrigeradores
Fogões e estufas
Aparelhos médicos
Transistores
Laser
Maser
TV
Impressão
Microprocessadores
Efeito fotoelétrico
Etc., etc.

18

ALGUMAS EQUAÇÕES DA FÍSICA

ALGUMAS das principais equações diferenciais da física não relativista são as seguintes, lembrando que a definição do operador de Laplace é a seguinte:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{18.1}$$

Equação de ondas de D'Alambert

$$\frac{\partial u}{\partial t^2} = a^2 \nabla^2 u. \tag{18.2}$$

Esta equação aplica-se em vibrações transversais de uma corda, em vibrações longitudinais de vigas, em oscilações elétricas, etc. (equação tipo hiperbólico)

Equação de Fourier ou da condução do calor

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u. \tag{18.3}$$

Ela se aplica na teoria da propagação do calor, em filtração de um líquido em meio poroso, em teoria da probabilidade, etc. (equação parabólica)

Equação de Laplace

$$\nabla^2 u = 0. \tag{18.4}$$

A equação aplica-se na teoria dos campos elétrico e magnético, na hidrodinâmica, na difusão, etc. (equação elítica).

Equação de Schrödinger

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{18.5}$$

Ela é aplicada na mecânica quântica não relativista e em processos néo-clássicos de física quântica.

Todas essas equações são clássicas no sentido de que não valem em teorias relativísticas.

Observações sobre a equação de Schrödinger

Em uma dimensão, a equação é

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi. \tag{18.6}$$

Quando há duas partículas, a equação é

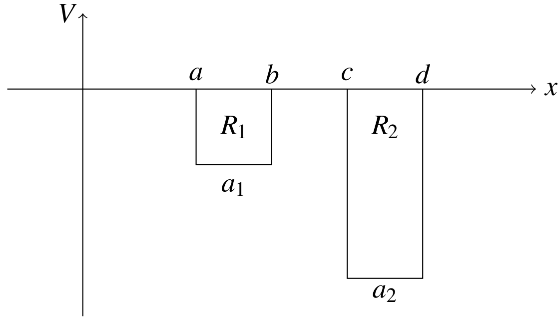
$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(r_1, r_2, t), \tag{18.7}$$

onde $\psi(r_1, r_2, t)$. A equação para várias partículas (em número finito) é análoga.

A equação precedente em uma dimensão espacial é

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2\psi}{\partial x_2^2} + V\psi. \tag{18.8}$$

Em especial, pode-se ter a seguinte situação:



Tem-se então: $a_1 \in R_1$ e $a_2 \in R_2$, sendo $R_1 \cap R_2 = \emptyset$. Logo, a_1 tem a propriedade de pertencer a R_1 e a_2 tem a propriedade de pertencer a R_2; porém, $a_1 \notin R_2$ e $a_2 \notin R_1$. Logo, a_1 e a_2 são distintos ($a_1 \neq a_2$), etc.

Daí, temos o seguinte:

1. As propriedades espaço-temporais podem separar objetos físicos, em particular, partículas quânticas. Quando isso é possível, há identidade definida, caso contrário, não há identidade definida.

2. Objetos físicos microscópicos, em dado fenômeno, podem "não apresentar uma identidade", podem ou não ter "quid". Por exemplo, partículas de mesma espécie separadas espaço-temporalmente possuem identidade devido à separabilidade. Por outro lado, Bose-Einstein e Fermi-Dirac evidenciam que, nas situações correspondentes, há objetos quânticos que parecem "ter perdido" a identidade.

3. Dado que a MQ constitui uma disciplina "aproximada", como pode ser base de uma ontologia? Ou será apenas uma teoria do "como se" de certos objetos? Qual a diferença, nesse sentido, entre a equação de Schrödinger e as demais acima?

4. Não existe dificuldade em se interpretar, em MQ, a identidade como relação de congruência (relação de equivalência compatível com os símbolos de operação e de predicado). Ela, então, seria uma espécie de "identidade disfarçada". Mais ainda: mesmo que se tenha a própria identidade, em geral não há possibilidade de distinção efetiva de

partículas idênticas em situações tais como nos aglomerados de Bose-Einstein e de Fermi-Dirac. Todavia, e isto é sumamente relevante, há aqui a possibilidade de escolha de cominhos alternativos no que se refere à MQ, como é claro. Um deles é o verdadeiro e o outro é falso? Haveria mundos alternativos em conexão com a identidade?

5. Quando há partículas nas partes A e B do espaço em certo instante t do tempo, as partículas que estão em A não são diferentes das que se encontram em B, isto é, nenhuma partícula de A é igual a outra em B? A resposta afirmativa ou a negativa não implica que existem partículas que não são iguais ? Ou, pelo menos, que não são congruentes? Em QM, a congruência não pode se reduzir à identidade? E a identidade não pode se reduzir à congruência?

Bibliography

[1] Aczel, P. H. G., Saturated intuitionistic theories. In Schmidt, H. A., Schütte, K. and Thiele, H. J. (eds.), *Contributions to Mathematical Logic*. Amsterdam: North Holland, 1968, pp. 1-11.

[2] Aharonov-Bohm effect, entry of Wikipedia. (This entry is a summary)

[3] Anouncement form Hans Dehmelt's laboratory in Seattle, 1984.

[4] Araki, H., *Mathematical Theory of Quantum Fields*, Oxford Univ. Press, 1999.

[5] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer, 1980.

[6] Arnold, V.I., *Métodos Matemáticos da Mecânica Clássica*, Mir, Moscou, 1987.

[7] Auyang, S. Y., *How is Quantum Field Theory Possible?*, Oxford Univ. Press, 2008.

[8] Baggott, J., *The Quantum Story*, Oxford Univ. Press, 2011.

[9] Bell, J. S., *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, 1987.

[10] Beth, E. W., *The Foundations of Mathematics*. Amsterdam: North Holland, 1959.

[11] Béziau, J. -Y., From consequence operator to universal logic: a survey of general abstract logic. In Béziau, J. -Y. (ed.), *Logica Universalis*. Birkhauser, 2006, pp. 3-17.

[12] Birkhoff, G., *Lattice Theory*. AMS, 3rd. ed., 1967.

[13] Bourbaki, N., *Theory of Sets*. Hermann & Addison-Wesley, 1968.

[14] Brignole, D. and da Costa, N. C. A., On supernormal Ehresmann-Dedecker Universes, *Mathematisch Zeitschrift*, 122, 342-350 1971.

[15] Church, A., *Introduction to Mathematical Logic*. Princeton Un. Press, 1956.

[16] Conway, J. and Kochen, S., The free will theorem, *Found. of Physics*, 36 (10), 1441-1473, 2006.

[17] Cubitt, T. S., Pérez-García, D. e Wolf, M., Um problema insolúvel, *Scientific American,* Novembro, 2018, 63-71.

[18] Curry, H. B., *Foundations of Mathematical Logic*. McGraw-Hill, 1963.

[19] Curry, H. B., *Leçons de Logique Algébrique*, Gauthiers-Vilars, 1952.

[20] Chaitim, G., N. da Costa and F. A. Doria, *Gödel's Way.* Taylor and Francis, London, 2012.

[21] da Costa, N. C. A., On the theory of inconsistent formal systems, *Notre Dame Journal of Formal Logic* XV (4), 497-510, 1974.

[22] da Costa N. C. A., *Ensaio sobre os Fundamentos da Lógica*. São Paulo: Hucitec, 1980

[23] da Costa, N. C. A., *Logiques Classiques et non Classiques*. Masson, Paris, 1997.

[24] da Costa, N. C. A, *O Conhecimento Científico*. São Paulo: Discurso Editorial, 1977.

[25] da Costa, N. C. A., A lógica da identidade e a Mecânica Quantica, 2012, neste volume.

[26] da Costa, N. C. A., Opérations non monotones dans les treillis, *C. R. Acad. Sc. Paris* 263, 1966, pp.429-32.

[27] da Costa, N. C. A. Filtres er idéaux d'une algèbre C_n, *C. R. Acad. Sc. Paris* 264, 1967, pp.549-52.

[28] da Costa, N. C. A., α-models and systems T and T^*. *Notre Dame J. Formal Logic* 14, 1974, pp. 443-54.

[29] da Costa, N. C. A., *Logiques Classiques et Non Classiques*. paris: Masson, 1997.

[30] da Costa, N. C. A., Remarks on Generalized Galois Theory. Preprint, UFSC, 2006.

[31] da Costa, N.C.A., Remarques sur le système NF1, *C. R. Acad Sciences Paris*, 27 A (l971), 1140-1151.

[32] da Costa, N. C. A. and Béziau, J. -Y. Théorie de la valuation, *Logique et Analyse* 145-146, 1994, pp. 95-117.

[33] da Costa, N. C. A. and Rodrigues, A. A. M., Definability and invariance. Preprint, UFSC, 2005.

[34] da Costa, N. C. A. and V. S. Subrahmanian, Paraconsistent logics as a formalism for reasoning about inconsistent knowledge bases, *Artificial Intelligence in Medicine*, 1, 167-174, 1989.

[35] da Costa, N. C. A. and O. Bueno, Non reflexive logics, *Revista Brasileira de Filosofia*, 58 (232), 181-196, 2009.

[36] da Costa, N. C. A. and N. Grana, *Il Recupero dell'inconsistenza.* L'Orientale Editrice, Napoli, 2009.

[37] da Costa, N. C. A., O. Bueno and S. French, The logic of pragmatic truth, *Journal of Philosophical Logic.* 27, 603-620, 1998.

[38] da Costa, N. C. A., D. Krause and O. Bueno, Paraconsistent logics and paraconsistency. *Handbook of the Philosophy of Science*. Philosophy of Logic, Dale Jacquette editor, Elsevier, 2007, pp. 791-911.

[39] da Costa, N. C. A. and S. French, Pragmatic truth and the logic of induction, *British Journal for the Philosophy of Science*, 40, 333-356, 1989.

[40] da Costa, N. C. A. and S. French, On Russell's principle of induction, *Synthese* 86, 285-295, 1991.

[41] da Costa, N. C. A. and S. French, *Science and Partial Truth.* Oxford University Press, 2003.

[42] da Costa, N. C. A. and F. A. Doria, *On the Foundations of Science.* Coppe, Rio de Janeiro, 2008.

[43] da Costa, N. C. A. and Doria, F. A., The Chaitin-Thorp theorem, to appear.

[44] da Costa, N. C. A., and C. de Ronde, The paraconsistent logic of quantum superpositions, *Found. of Physics*, 43, 845-858, 2013.

[45] da Costa, N. C. A., and J. Kouneiher, *Cohomological foundations of quantum-physics*, preprint, 2017.

[46] da Costa, N. C. A. and Joseph Kouneiher, Cohomologie, Homologie et l'Universel en Physique Mathématique, preprint, 2017.

[47] da Costa, N.C.A. e de Ronde, Non-reflexive logical foundations for quantum mechanics, *Foundations of Physics*, 44 (2014), 1369-1380.

[48] da Costa, N.C.A. e de Ronde, On the foundations of quantum mechcanics, a aparecer, 2014.

[49] da Costa, N.C.A. e de Ronde, Revisiting the applicability of metaphysical identity in quantum mechanics, a aparecer, 2016.

[50] Dzik, W. The existence of Lindenbaum extensions is equivalent to the axiom of choice, *Reports on Mathematical Logic* 13, 1981, pp. 29-31.

[51] Falkenburg, B., *Particle Metaphysics: A Critical Account of Sub-Atomic Reality*, Springer, 2007.

[52] Farah, E., *Algumas Proposições Equivalentes ao Axioma da Escolha*. Curitiba: Editora da UFPR, 1994.

[53] Fraenkel, A. A., *Abstract Set Theory*, North-Holland, 1953.

[54] Fraenkel, A. A. e Bar-Hillel, Y., *Foundations of Set theory,* North-Holland, 1958.

[55] Franzén, T., *Godel's Theorem*, Peters, 2005.

[56] French, S. and D. Krause, *Identity in Physics: A Historical, Philosophical and Formal Analysis*. Oxford University Press, 2006.

[57] Gleason, A. M., Measures on the closed subspaces of a Hilbert space, *Indiana Univ., Mathematics Journal*, 6 (4), 885-893, 1957.

[58] Gottwald, S., *Fuzzy Sets and Fuzzy Logic*, Vieweg 1995.

[59] Griffiths, D., *Introduction to Quantum Mechanics,* Pearson-Prentice Hall, 2005.

[60] Haag, R., *Local Quantum Physics*, Springer, Berlin, 1992.

[61] Haag, R., *Local Quantum Physics: Fields, Particles, Algebras*, Springer, 1996.

[62] Hanson, R. e Shalm, K., Uma estranha realidade, *Scientific American*, fevereiro, 2019, 37-43.

[63] Hawking S., Gödel and the end of physics, July 20, 2002.www.dampt.ac.uk

[64] Hawking, S. and Penrose, R., *The Nature of Space and Time*. Princeton University Press, 1996.

[65] Heyting, A., *Les Fondements des Mathématiques, Intuitionisme, Théorie de la Démonstration*, Gauthier-Villars, 1955.

[66] Heyting, A. *Intuitionism: An Introduction*. North-Holland, 1956.

[67] Hilbert, D., Mathematische Probleme, *Archiv. Der Mathematik und Physik*, 3dser., vol 1, 44-63 and 213-237 1901.

[68] Hughes, R. I. G., *The Structure and Interpretation of Quantum Mechanics*, Harvard University Press, 1992.

[69] https://physorg/news/2019-geometry-electron.html (Trabalhos de Dominik Zumbühl e Daniel Loss, The geometry of the electron determined for the first time, The University of Basel.)

[70] Isham, C. J., *Lectures on Quantum Theory*, Imperial College Press, London, 1995.

[71] Isham, C. J., *Quantum Theory*, Imperial College Press, 2008. (Constitui uma excelente introdução à MQ.)

[72] Jaki, S. L., *The Relevance of Physics*, Chicago Press, 1966.

[73] Jaki S. L., A late awakening to Gödel in physics, www.sljaki.com/jakigodel.pdf

[74] Ji, L. e Liu, K., Shing-Tung Yau : a Manifold Man of Mathematics, *Proc. Geometric Analysis: Present and Future*, Harvard University, 2008.

[75] Jauch, J. M., *Foundations of Quantum Mechanics*, Addison-Wesley, 1968.

[76] Kleene, S. C., *Introduction to Metamathematics*. Van Nostrand, 1952.

[77] Kleene, S. C., Disjunction and existence under implication in elementary intuitionistic formalisms. *J. Symbolic Logic* 27, 1962, pp. 11-18.

[78] Kleene, S. C., On the interpretation of intuiotionistic number theory, *J. Symbolic Journal* 10, 1945, pp. 109.

[79] Kolmogorov, A. N., On the principle of excluded middle. In van Heijenoort, J. (ed.), *From Frege to Gödel*. Harvard Un. Press, 1967, pp. 416-37.

[80] Kelley, J. L., *General Topology*. Van Nostrand, 1955.

[81] Kochen, S. and Specker, E., The problem of hidden variables in quantum mechanics, *Journal of Math. And Mech*, 17, 59-87, 1967.

[82] Krause, D., On quasi-set theory, *Notre Dame J. Form. Logic*, 33, 402-411, 1992.

[83] Kuhlmann M., *The Ultimate Constituents of the Material World*, Ontos Verlag, 2010.

[84] Kuratowski, K. and Mostowski, A., *Set Theory*. North-Holland, 1968.

[85] Manin Yu. I., *A Course in Mathematical Logic for Mathematicians*, Springer, 2010, sobretudo a parte 12, “Quantum Logic”, muito importante para todos os interessadoas em fundamentos da QM.

[86] Martin, N. M. and Pollard, S., *Closure Spaces and Logic*. Kluwer, 1996.

[87] Mendelson, E., *Introduction to Mathematical Logic*, second edition, Van Nostrand, 1979.

[88] Mikenberg, I., N. C. A. da Costa and R. Chuaqui, Pragmatic truth and approximation of truth, *Journal of Symbolic Logic*, 1986, 201-221.

[89] Moore, G. H., *Zermelo's Axiom of Choice. Its Origins, Developement, and Influence*. Springer, 1982.

[90] Pitowski, I., Betting on the outcomes of measurements: a Bayesian theory of quantum probability, *Studies in Hist. and Philos. of Modern Physics*, 34 (3) 395-414 (2013).

[91] Rescher, N., *Many-Valued Logics*. McGraw-Hill, 1969.

[92] Rosser, J. B., *Logic for Mathematicians.*. McGraw-Hill, 1953.

[93] Russell, Bertrand, *The Art of Philosophizing*, Philosophical Library, 1968.

[94] Shoenfield, J. R., *Mathematical Logic*. Addison-Wesley, 1967.

[95] Sobre as obras de Haroche e Wineland, consultar: The Nobel Prize in Physics 2012, The Royal Swedish Academy of Sciences, http://kva.se

[96] Styer, D. F., and others, Nine formulations of quantum mechanics, *Am. J. Physics* 70 (3), 288-297, 2002.

[97] Stromberg, K., The Banach-Tarski paradox, *Am. Math. Monthly*, MAA, 86 (3), 1979, 151-161.

[98] Suszko, R., Concerning of logical schemes, the notion of logical calculus and the role of consequence relation. *Studia Logica* 11, 1961, pp.189-215.

[99] Tarski, A., *Logic, Semantics, Metamathematics*, Hackett, 1983.

[100] “The Banach-Tarski paradox”, Wikipedia.

[101] The Nobel Prize in Physics 2012, The Royal Academy of Sciences. http://kva.sve.

[102] von Neumann, J., *Mathematical Foundations of Quantum Mechanics,* Princeton University Press edição de 1955.

[103] Wapner, L. M., *The Pea and the Sun: a Mathematical Paradox*, A.K. Peters, 2005.

[104] Wick, D., *The Infamous Boundary*, Copernicus, 1995. (No capítulo 15, trata detalhadamente do trabalho de Hans Dehmelt.)

[105] Witten, E., A new proof of the positive energy theorem, *Communications in Mathematical Physics* 80 (1981), 381-402.

[106] Wójcicki, R., *Theory of Logical Calculi*, Kluwer, 1988.

[107] Wójcicki, R., *Lectures of Propositional Calsuli*. Polish Acad. of Sciences, 1984.

[108] Yau, Shing-Tung, e Nadis, S., *The Shape of Inner Space*, Basic Books, New York, 2010.